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Extremal values of eigenvalues of Sturm–Liouville operators with potentials in L^1 balls

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A B S T R A C T

This paper is a continuation of Zhang [M. Zhang, Continuity in weak topology: Higher order linear systems of ODE, Sci. China Ser. A 51 (2008) 1036–1058; M. Zhang, Extremal values of smallest eigenvalues of Hill's operators with potentials in L^1 balls, J. Differential Equations 246 (2009) 4188–4220]. Given a potential $q \in L^p([0,1],\mathbb{R})$, $p \in [1,\infty]$. We use $\lambda_m(q)$ to denote the mth Dirichlet eigenvalue of the Sturm–Liouville operator with potential $q(t)$, where $m \in \mathbb{N}$. The minimal value $\mathbf{L}_{m,p}(r)$ and the maximal value $\mathbf{M}_{m,p}(r)$ of $\lambda_m(q)$ with potentials q in the L^p ball of radius r are well defined. In this paper, we will exploit the continuity of $\lambda_m(q)$ in q with weak topologies and the variational method to give characterizations of $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ when $p \in (1,\infty)$. By using the limiting approach as $p \downarrow 1$, we find that the most important extremal values $\mathbf{L}_{m,1}(r)$ and $\mathbf{M}_{m,1}(r)$ can be evaluated explicitly using some elementary functions of r. The corresponding extremal problems for Neumann eigenvalues and some periodic eigenvalues will be reduced to $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$.

1. Introduction

This paper is a continuation of recent works by Zhang [21,22]. We aim at giving the extremal values of eigenvalues of Sturm–Liouville operators with potentials in L^1 balls.

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Given a (real) potential q in the Lebesgue space $\mathcal{L}^p := L^p([0,1],\mathbb{R})$, $1 \leqslant p \leqslant \infty$. The eigenvalue problem

$$x'' + \big(\lambda + q(t)\big)x = 0 \tag{1.1}$$

with the Dirichlet boundary condition

$$x(0) = x(1) = 0 \tag{1.2}$$

has a sequence of eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_m(q) < \cdots, \quad \lambda_m(q) \to +\infty.$$

See [2,18,19]. We use $\|q\|_p := \|q\|_{L^p[0,1]}$ to denote the L^p norm for $q \in \mathcal{L}^p$. Given $r \in [0,\infty)$. Let

$$B_p[r] := \big\{q \in \mathcal{L}^p:\ \|q\|_p \leqslant r\big\}$$

be the ball of the radius r centered at 0 in the space $\mathcal{L}^p$. In this paper, we will study the following extremal values

$$\mathbf{L}_{m,p}(r) := \inf\big\{\lambda_m(q):\ q \in B_p[r]\big\}, \qquad \mathbf{M}_{m,p}(r) := \sup\big\{\lambda_m(q):\ q \in B_p[r]\big\}. \tag{1.3}$$

Here the parameters are

$$m \in \mathbb{N}, \qquad p \in [1,\infty], \qquad r \in [0,\infty). \tag{1.4}$$

Since $\lambda_m(0) = (m\pi)^2$, one has

$$\mathbf{L}_{m,p}(0) = \mathbf{M}_{m,p}(0) = (m\pi)^2. \tag{1.5}$$

These extremal values $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ are well defined, finite for all (m,p,r) as in (1.4). To see this, let us consider $\lambda_m(q)$ as a (nonlinear) functional from $\mathcal{L}^p$ to $\mathbb{R}$. It is well known that $\lambda_m : (\mathcal{L}^p, \|\cdot\|_p) \to \mathbb{R}$ is continuously differentiable. See, for example, [6,18]. A deep result proved by Zhang [21] very recently shows that eigenvalues have stronger dependence on potentials.

Theorem 1.1. *(Zhang [21].) Given $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. Then*

$$\big(\mathcal{L}^p, w_p\big) \to \mathbb{R}, \qquad q \to \lambda_m(q)$$

is continuous. Here w_p is the topology of weak convergence in the space $\mathcal{L}^p$ for $1 \leqslant p < \infty$ and w_∞ is the topology of weak convergence in the space $\mathcal{L}^\infty$.*

Other kinds of eigenvalues of Sturm–Liouville operators possess the same continuity [21]. For some related continuity results of solutions and eigenvalues in weak topologies, see also [4,10,11,15,17].

Let $1 < p \leqslant \infty$. Since $B_p[r] \subset (\mathcal{L}^p, w_p)$ is sequentially compact for any $r \geqslant 0$, an immediate consequence of Theorem 1.1 is that both $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ can be attained by some potentials in $B_p[r]$ and therefore are finite. However, when $p = 1$, the finiteness of $\mathbf{L}_{m,1}(r)$ and $\mathbf{M}_{m,1}(r)$ cannot be deduced from Theorem 1.1 directly because L^1 balls are no longer compact even in the weak topology w_1. In order to obtain the finiteness of $\mathbf{L}_{m,1}(r)$, let us recall a deep result of Zhang [22]. We use $\lambda_0^N(q)$ to denote the smallest or the zeroth Neumann eigenvalue of (1.1) for $q \in \mathcal{L}^p$.

Theorem 1.2. *(Zhang [22, Theorem 6.5].) For any $r \in [0,\infty)$, one has*

$$\mathbf{L}_{0,1}^N(r) := \inf\bigl\{\lambda_0^N(q)\colon\ q \in B_1[r]\bigr\} = \hat{\mathbf{Z}}_0^{-1}(r). \tag{1.6}$$

Here $\hat{\mathbf{Z}}_0(x)$ is the following elementary function

$$\hat{\mathbf{Z}}_0(x) := \sqrt{-x}\tanh(\sqrt{-x}), \quad x \in (-\infty, 0],$$

which is a decreasing diffeomorphism from $(-\infty, 0]$ onto $[0,\infty)$.

Since

$$\lambda_m(q) > \lambda_0^N(q), \quad \forall q \in \mathcal{L}^1,\ \forall m \in \mathbb{N},$$

it follows from (1.5) and (1.6) that

$$\mathbf{L}_{m,1}(r) \in \bigl[\mathbf{L}_{0,1}^N(r), \mathbf{L}_{m,1}(0)\bigr] = \bigl[\hat{\mathbf{Z}}_0^{-1}(r), (m\pi)^2\bigr].$$

Consequently, $\mathbf{L}_{m,1}(r)$ is always finite for any $m \in \mathbb{N}$ and $r \geqslant 0$.

Note from [22, formula (6.6)] that the maximal values of $\lambda_0^N(q)$ in $B_p[r]$ are trivially

$$\mathbf{M}_{0,p}^N(r) := \sup\bigl\{\lambda_0^N(q)\colon\ q \in B_1[r]\bigr\} = r, \quad p \in [1,\infty].$$

In order to see that $\mathbf{M}_{m,1}(r)$ is also well defined, let us recall some asymptotical distribution results of large eigenvalues like

$$\lambda_n(q) = (n\pi)^2 + O(1) \quad \text{as } n \to +\infty.$$

See, e.g., [8,11]. In fact, this asymptotical result is uniform for potentials q in any bounded subset of $(\mathcal{L}^p, \|\cdot\|_p)$, including the L^1 case. For the L^2 case, such a uniformity can be found in [11]. For the L^1 case, it can be obtained by a modification of the proof there. The uniformity above means that for any $r \geqslant 0$, there exist $C_r \gg 1$ and $n_r \gg 1$ such that

$$\bigl|\lambda_n(q) - (n\pi)^2\bigr| \leqslant C_r, \quad \forall q \in B_1[r],\ n \geqslant n_r.$$

Hence

$$(m\pi)^2 = \mathbf{M}_{m,1}(0) \leqslant \mathbf{M}_{m,1}(r) \leqslant C_r + \pi^2\bigl(\max\{m, n_r\}\bigr)^2 < \infty, \quad m \in \mathbb{N}.$$

Thus all extremal values of (1.3) are finite and are well defined.

As noted in [22], the most important extremal values of (1.3) are those in L^1 balls. The main aim of this paper is to give an explicit construction of $\mathbf{L}_{m,1}(r)$ and $\mathbf{M}_{m,1}(r)$. Define $R_m : [0,\infty) \to [0,\infty)$ by

$$R_m(x) = x/m^2.$$

For the minimal values $\mathbf{L}_{m,p}(r)$, the main results are as follows.

Theorem 1.3. *Let* (m, p, r) *be as in* (1.4)*. There holds the following amplification equality between extremal values of higher order and the principal eigenvalues*

$$\mathbf{L}_{m,p}(r) = R_m^{-1} \circ \mathbf{L}_{1,p} \circ R_m(r). \tag{1.7}$$

Define the function $\mathbf{Z}_1 : (-\infty, \pi^2] \to [0, \infty)$ by

$$\mathbf{Z}_1(x) = \begin{cases} 2\sqrt{-x}\coth(\sqrt{-x}/2) & \text{for } x \in (-\infty, 0), \\ 4 & \text{for } x = 0, \\ 2\sqrt{x}\cot(\sqrt{x}/2) & \text{for } x \in (0, \pi^2]. \end{cases} \tag{1.8}$$

Then $\mathbf{Z}_1$ is a decreasing diffeomorphism mapping $(-\infty, \pi^2]$ onto $[0, \infty)$.

Theorem 1.4. *Let* $p = 1$*. One has, for all* $r \in [0, \infty)$*,*

$$\mathbf{L}_{1,1}(r) = \mathbf{Z}_1^{-1}(r). \tag{1.9}$$

By (1.7)–(1.9), we have obtained $\mathbf{L}_{m,1}(r)$ for all $m \in \mathbb{N}$.

The results for the maximal values $\mathbf{M}_{m,p}(r)$ are as follows. Define $\mathbf{Y}_1 : [0, \infty) \to [\pi^2, \infty)$ by

$$\mathbf{Y}_1(x) := \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4x}\right)^2. \tag{1.10}$$

Then $\mathbf{Y}_1$ is an increasing diffeomorphism mapping $[0, \infty)$ onto $[\pi^2, \infty)$.

Theorem 1.5.

(i) *Let* (m, p, r) *be as in* (1.4)*. There holds the following amplification equality*

$$\mathbf{M}_{m,p}(r) = R_m^{-1} \circ \mathbf{M}_{1,p} \circ R_m(r). \tag{1.11}$$

(ii) *Let* $p = 1$*. One has, for all* $r \in [0, \infty)$*,*

$$\mathbf{M}_{1,1}(r) = \mathbf{Y}_1(r). \tag{1.12}$$

By (1.10)–(1.12), we have obtained $\mathbf{M}_{m,1}(r)$ for all $m \in \mathbb{N}$.

For the case $p \in (1, \infty)$, though $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ can be expressed using singular integrals, they are not elementary functions of r. However, for the most important case $p = 1$, Theorems 1.3–1.5 show that all $\mathbf{L}_{m,1}(r)$ and $\mathbf{M}_{m,1}(r)$ are elementary functions of r.

The general program of proofs in this paper is like [22] where the extremal values of the smallest periodic eigenvalues and the smallest Neumann eigenvalues with potentials in L^1 balls are considered. Roughly speaking, the proof of Theorem 1.4 consists of the following three steps.

- For $p \in (1, \infty)$, it follows from Theorem 1.1 that $\mathbf{L}_{m,p}(r)$ has minimizers in $B_p[r]$, which are actually on $S_p[r]$.
- By the Lagrangian multiplier method, one can deduce the critical equations for minimizers of $\mathbf{L}_{m,p}(r)$ and yield a representation for $\mathbf{L}_{m,p}(r)$ with $p \in (1, \infty)$.
- The final result (1.9) is obtained from a careful examination on the limit $\lim_{p \downarrow 1} \mathbf{L}_{m,p}(r)$, which is just $\mathbf{L}_{m,1}(r)$.

The structure of this paper is as follows.

In Section 2, we will give some basic properties on eigenvalues and extremal values, following Zhang [22]. Due to some topological fact on L^p balls [22], one has the limiting equality for $\mathbf{L}_{m,1}(r)$

$$\mathbf{L}_{m,1}(r) = \lim_{p\downarrow 1} \mathbf{L}_{m,p}(r).$$

Similar limiting equality is also true for $\mathbf{M}_{m,1}(r)$. See Lemma 2.4. Let us introduce L^p spheres as

$$S_p[r] := \{q \in \mathcal{L}^p\colon\ \|q\|_p = r\} \subset B_p[r].$$

Due to basic properties of eigenvalues, the extremal values $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ in L^p balls are actually the same as those on the L^p spheres

$$\mathbf{L}_{m,p}(r) := \inf\{\lambda_m(q)\colon\ q \in S_p[r]\}, \qquad \mathbf{M}_{m,p}(r) := \sup\{\lambda_m(q)\colon\ q \in S_p[r]\} \tag{1.13}$$

for all (m, p, r) as in (1.4). See Lemma 2.6.

In Section 3, we use the variational method to characterize the minimal values $\mathbf{L}_{m,p}(r)$ with $p \in (1,\infty)$. The critical equations for $\mathbf{L}_{m,p}(r)$ have different nature for small radius and large radius. The exact zeros $\mathbf{R}_{m,p}$ of the functions $\mathbf{L}_{m,p}(\cdot)$ found by Zhang [20] can distinguish this. For the construction of these important radii $\mathbf{R}_{m,p}$, see also Section 2.2. Using these critical equations, together with some limiting analysis, amplification relation (1.7) of Theorem 1.3 will be proved at the end of Section 3.

In Section 4 we will study $\lim_{p\downarrow 1} \mathbf{L}_{1,p}(r)$ and complete the proof of Theorem 1.4. Though the basic idea is as in [22] for $\mathbf{L}^N_{0,p}(r)$, some considerations for $\mathbf{L}_{1,p}(r)$ are more delicate, because the critical equations for $\mathbf{L}_{1,p}(r)$ have some nature different from that for $\mathbf{L}^N_{0,p}(r)$. The proof of Theorem 1.4 will be given at the end of Section 4.2. In Section 4.3, the asymptotical formula for $\mathbf{L}_{m,p}(r)$ with large r will be given. See Corollary 4.8.

In Section 5, we will derive the critical equations for the maximal values $\mathbf{M}_{m,p}(r)$ with $p \in (1,\infty)$. The amplification (1.11) can be obtained from the corresponding critical equations. However, in studying $\lim_{p\downarrow 1} \mathbf{M}_{1,p}(r)$, we will adopt an approach which is completely different from that in Section 4. This is caused by the facts that $\mathbf{M}_{m,1}(r)$ can be attained by some potentials in $B_1[r]$, while $\mathbf{L}_{m,1}(r)$ cannot be attained by any potential in $B_1[r]$. See Remark 4.7 and Corollary 5.8. Roughly speaking, result (1.12) is obtained by finding the limiting equation of the critical equations of $\mathbf{M}_{1,p}(r)$ as $p \downarrow 1$. The detailed proof of Theorem 1.5 is given in Section 5.2.

Recall from [19] that there are some close relations between several kinds of eigenvalues of Sturm–Liouville operators, including the Dirichlet, the Neumann, periodic and anti-periodic eigenvalues. In Section 6, we will exploit these relations to show that some extremal problems of other eigenvalues can be completely reduced to $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ of the Dirichlet eigenvalues. For example, if we use $\mathbf{L}^N_{m,p}(r)$ and $\mathbf{M}^N_{m,p}(r)$, $m \in \mathbb{N}$, to denote the corresponding extremal values of the mth Neumann eigenvalues $\lambda^N_m(q)$, it will be proved that

$$\mathbf{L}^N_{m,p}(r) = \mathbf{L}_{m,p}(r), \qquad \mathbf{M}^N_{m,p}(r) = \mathbf{M}_{m,p}(r).$$

Some extremal problems of higher-order periodic and anti-periodic eigenvalues of Hill's operators can also be reduced to $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$. See Theorem 6.1 and Corollary 6.2.

Combining with the extremal values found in [22] for the smallest periodic eigenvalues and the smallest Neumann eigenvalues, we have given a fairly complete construction of extremal values for eigenvalues of Sturm–Liouville operators with potentials in L^1 balls. Some open problems on extremal values of periodic eigenvalues will be imposed in Section 6.

2. General results on eigenvalues and extremal values

2.1. Eigenvalues, eigen-functions and extremal values

Most of the results of this subsection are analogous to that for the smallest periodic eigenvalues of Hill's operators as in [22, Section 2]. We will give only the statements and necessary proofs. For details, one may refer to [22].

Given $q \in \mathcal{L}^p$. The following characterization on the mth eigenvalue $\lambda_m(q)$ using nodes of eigen-functions is a basic result in spectral theory [11,18].

Lemma 2.1. *Let $\lambda \in \mathbb{R}$ be an eigenvalue of* (1.1)–(1.2) *and $E(t)$ be an eigen-function associated with λ. Then λ is the mth eigenvalue $\lambda_m(q)$ if and only if $E(t)$ has precisely $(m+1)$ zeros in the interval* $[0,1]$, *including* 0 *and* 1.

One has the following comparison or monotonicity results for eigenvalues [19].

Lemma 2.2. *Let $q_i \in \mathcal{L}^p$. If $q_1(t) \geqslant q_2(t)$ a.e. t, then $\lambda_m(q_1) \leqslant \lambda_m(q_2)$. If, furthermore, the strict inequality $q_1(t) > q_2(t)$ holds on a subset of* $[0,1]$ *of positive measure, the strict inequality $\lambda_m(q_1) < \lambda_m(q_2)$ is true.*

Suppose that $q \in \mathcal{L}^\infty$. By Lemma 2.2, one has

$$(m\pi)^2 - \operatorname{ess\,sup} q = \lambda_m(\operatorname{ess\,sup} q) \leqslant \lambda_m(q) \leqslant \lambda_m(\operatorname{ess\,inf} q) = (m\pi)^2 - \operatorname{ess\,inf} q.$$

From these, one can obtain

$$\mathbf{L}_{m,\infty}(r) = \lambda_m(+r) = (m\pi)^2 - r, \qquad \mathbf{M}_{m,\infty}(r) = \lambda_m(-r) = (m\pi)^2 + r. \tag{2.1}$$

By the Hölder inequality, one has $B_{p_1}[r] \supset B_{p_2}[r]$ for all $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$. From the definition, $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ possess the following monotonicity.

Lemma 2.3. *Let m and r be fixed. Then $\mathbf{L}_{m,p}(r)$ is non-decreasing and $\mathbf{M}_{m,p}(r)$ is non-increasing in $p \in [1,\infty]$.*

Like [22, Lemma 2.2], one has the following limiting equalities.

Lemma 2.4. *One has the following limiting equalities*

$$\mathbf{L}_{m,1}(r) = \lim_{p\downarrow 1} \mathbf{L}_{m,p}(r) = \inf_{p\in(1,\infty)} \mathbf{L}_{m,p}(r), \tag{2.2}$$

$$\mathbf{M}_{m,1}(r) = \lim_{p\downarrow 1} \mathbf{M}_{m,p}(r) = \sup_{p\in(1,\infty)} \mathbf{M}_{m,p}(r). \tag{2.3}$$

Proof. We only prove (2.2) because (2.3) is similar. By the monotonicity of $\mathbf{L}_{m,p}(r)$ in p in Lemma 2.3, we know that the second equality of (2.2) holds.

Note that $B_p[r] \subset B_1[r]$ implies that

$$\mathbf{L}_{m,1}(r) \leqslant \mathbf{L}_{m,p}(r), \quad \forall p \in (1,\infty).$$

Taking the limit as $p \downarrow 1$, we get

$$\mathbf{L}_{m,1}(r) \leqslant \lim_{p\downarrow 1} \mathbf{L}_{m,p}(r).$$

On the other hand, for any $q \in B_1[r]$, it follows from [22, Lemma 2.1] that there exists $q_p \in B_p[r]$ such that $\|q_p - q\|_1 \to 0$ as $p \downarrow 1$. Thus

$$\lambda_m(q) = \lim_{p\downarrow 1} \lambda_m(q_p) \geqslant \lim_{p\downarrow 1} \mathbf{L}_{m,p}(r).$$

Taking the infimum over $q \in B_1[r]$, we get

$$\mathbf{L}_{m,1}(r) \geqslant \lim_{p\downarrow 1} \mathbf{L}_{m,p}(r).$$

Hence we have (2.2). $\square$

Lemma 2.5. *Assume that*

$$m \in \mathbb{N}, \qquad p \in (1,\infty), \qquad r \in (0,\infty). \tag{2.4}$$

(i) *The minimal value* $\mathbf{L}_{m,p}(r)$ *can be attained by some* $\hat{q} \in B_p[r]$*, called a minimizer. Moreover,*

$$\hat{q}(t) \geqslant 0 \quad \text{a.e. } t \in [0,1], \qquad \|\hat{q}\|_p = r. \tag{2.5}$$

(ii) *The maximal value* $\mathbf{M}_{m,p}(r)$ *can be attained by some* $\check{q} \in B_p[r]$*, called a maximizer. Moreover,*

$$\check{q}(t) \leqslant 0 \quad \text{a.e. } t \in [0,1], \qquad \|\check{q}\|_p = r. \tag{2.6}$$

The existence of minimizers and maximizers follows from Theorem 1.1 and compactness of $B_p[r]$ in weak topology w_p. Properties (2.5) and (2.6) follow from comparison results of Lemma 2.2. The detailed proof is like that for Lemma 2.9 of [22].

Lemma 2.6. *For all* (m, p, r) *as in* (1.4)*, one has the equalities described in* (1.13)*.*

Proof. For $p \in (1,\infty)$, equalities in (1.13) have been obtained in (2.5) and (2.6). For $p = \infty$, the equalities are trivial from (2.1). In case $p = 1$, one can deduce (1.13) using the corresponding results for $p \in (1,\infty)$ and the limiting equalities (2.2) and (2.3). $\square$

For the case $p = 2$, results in (1.13) have been observed in [11] using the basic fact that $\lambda_m(q+c) = \lambda_m(q) - c$ for all $c \in \mathbb{R}$.

Next we consider $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ as functions of radius r.

Lemma 2.7. *Let* $p \in [1,\infty]$ *and* $m \in \mathbb{N}$*. As functions of r,*

$$[0,\infty) \to \left(-\infty, (m\pi)^2\right], \qquad r \to \mathbf{L}_{m,p}(r)$$

is a decreasing homeomorphism, while

$$[0,\infty) \to \left[(m\pi)^2, \infty\right), \qquad r \to \mathbf{M}_{m,p}(r)$$

is an increasing homeomorphism.

Note that the continuity of $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ in r can be simply deduced from the continuity of $\lambda_m(q)$ in $q \in (\mathcal{L}^p, \|\cdot\|_p)$. The most important ingredient of Lemma 2.7 is that $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ are strictly monotone in r. For the case $p = \infty$, this is trivial from (2.1). For the case $p \in (1,\infty)$, as $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ have minimizers and maximizers respectively, one can see that the proofs of Lemmas 2.10 and 2.11 of [22] also apply to the present case. Hence one can actually obtain stronger monotonicity, like result (2.11) of [22]. For the case $p = 1$, one can obtain from the results for $p \in (1,\infty)$ and the limiting equalities (2.2) and (2.3). For more details, see [22, Section 2].

2.2. Sobolev inequalities and zeros of minimal functions

Let $m \in \mathbb{N}$ and $p \in [1,\infty]$ be fixed. By Lemma 2.7, $\mathbf{L}_{m,p}(r)$ is strictly decreasing in $r \in [0,\infty)$. Note that $\mathbf{L}_{m,p}(0) = (m\pi)^2$ and $\lim_{r\uparrow\infty} \mathbf{L}_{m,p}(r) = -\infty$. Hence there exists the unique $\mathbf{R}_{m,p} \in (0,\infty)$ such that $\mathbf{L}_{m,p}(r) = 0$ at $r = \mathbf{R}_{m,p}$. These zeros $\mathbf{R}_{m,p}$ have been found explicitly by Zhang [20] in the study of non-degeneracy of the p-Laplacian. See also [23] for their role in the stability of Hill's equations.

Consider the following Sobolev inequality

$$C\|u\|_\gamma^2 \leqslant \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,1),$$

where the exponent $\gamma \in [1,\infty]$. The optimal Sobolev constant is denoted by $\mathbf{K}(\gamma)$. Explicitly,

$$\mathbf{K}(\gamma) = \inf_{u\in H_0^1(0,1),\, u\neq 0} \frac{\|u'\|_2^2}{\|u\|_\gamma^2} = \begin{cases} \frac{2\pi}{\gamma}\left(\frac{2}{\gamma+2}\right)^{1-2/\gamma}\left(\frac{\Gamma(1/\gamma)}{\Gamma(1/2+1/\gamma)}\right)^2 & \text{for } 1 \leqslant \gamma < \infty, \\ 4 & \text{for } \gamma = \infty. \end{cases}$$

See, for example, [14]. Here $\Gamma(\cdot)$ is the Gamma function of Euler. As a function of $\gamma \in [1,\infty]$, $\mathbf{K}(\gamma)$ is continuous and is strictly decreasing in γ.

In the terminology of the minimal values $\mathbf{L}_{m,p}(r)$ of this paper, the results of [20] can be stated as follows.

Lemma 2.8. *(Zhang [20].)*

(i) *Let $p \in [1,\infty]$ and $p^* = p/(p-1) \in [1,\infty]$ be the conjugate exponent of p. Then*

$$\mathbf{L}_{1,p}(r) \geqslant \pi^2\big(1 - r/\mathbf{K}(2p^*)\big), \quad \forall r \in \big[0, \mathbf{K}(2p^*)\big]. \tag{2.7}$$

(ii) *The zeros $\mathbf{R}_{m,p}$ are given by*

$$\mathbf{R}_{m,p} = m^2\mathbf{K}(2p^*), \quad m \in \mathbb{N},\ p \in [1,\infty]. \tag{2.8}$$

In particular, $\mathbf{R}_{m,1} = m^2\mathbf{K}(\infty) = 4m^2$ for all $m \in \mathbb{N}$.

The minimal value $\mathbf{L}_{1,p}(r)$ and its lower bound $\pi^2(1 - r/\mathbf{K}(2p^*))$ in (2.7) share the same zero, i.e., $r = \mathbf{K}(2p^*)$. The zeros $\mathbf{R}_{m,p}$ of $\mathbf{L}_{m,p}(r)$ are found using this and the node structure of Dirichlet eigenfunctions. After we find precise values of $\mathbf{L}_{1,p}(r)$ in this paper, a comparison to its lower bound $\pi^2(1 - r/\mathbf{K}(2p^*))$ can be given. Results (2.7) and (2.8) have been extended to the so-called p-Laplacian [20].

3. Variational approach to minimal values $\mathbf{L}_{m,p}(r)$

In this section, we always assume that (m, p, r) is as in (2.4).

3.1. Critical equations

From (1.13) and (2.5), one has

$$\mathbf{L}_{m,p}(r) = \min\{\lambda_m(q):\ q \in S_p[r]\}. \tag{3.1}$$

As a functional of potentials in the space $(\mathcal{L}^p, \|\cdot\|_p)$, the mth Dirichlet eigenvalue $\lambda_m(q)$ is continuously differentiable in $q \in (\mathcal{L}^p, \|\cdot\|_p)$ [6,9,18], because $\lambda_m(q)$ is simple. See also [17] for a simpler treatment and its generalization to the p-Laplacian. Let $E(t) = E(t;q)$ be an eigen-function associated with $\lambda_m(q)$. Then the differential of $\lambda_m(q)$ at q is given by

$$\partial_q \lambda_m(q) = -E^2(t;q)/\|E(\cdot;q)\|_2^2. \tag{3.2}$$

We write the constraint $q \in S_p[r]$ of (3.1) as the following equation

$$\|q\|_p^p = \int_0^1 |q(t)|^p\, \mathrm{d}t = r^p. \tag{3.3}$$

Note that the functional $q \to \|q\|_p^p$ is continuously differentiable in $q \in (\mathcal{L}^p, \|\cdot\|_p)$ with the differential

$$\partial_q \|q\|_p^p = p|q(t)|^{p-2} q(t). \tag{3.4}$$

Suppose that $q \in S_p[r]$ is a minimizer of problem (3.1). By the Lagrangian multiplier method [13], we obtain from (3.2) and (3.4) the following equality

$$E^2(t) = c_0 |q(t)|^{p-2} q(t), \tag{3.5}$$

where $c_0 \neq 0$ is the multiplier.

We are going to deduce from (3.5) the critical equation for $q(t)$. As an eigen-function, $E(t)$ satisfies

$$E'' + \big(\lambda_m(q) + q(t)\big)E = 0 \tag{3.6}$$

and boundary condition (1.2). Moreover, $E(t)$ has exactly $(m+1)$ zeros in $[0,1]$, including 0 and 1. For definiteness, we always take $E(t)$ so that $E'(0) > 0$. As $E(t)$ has only non-degenerate zeros and the minimizer $q(t)$ is non-negative, one sees from (3.5) that $c_0 > 0$ and $q(t) \geqslant 0$ for all t. Moreover, $q(t)$ has the same zeros as $E(t)$. By (3.5) and (3.6), both $E(t)$ and $q(t)$ are actually C^∞ functions.

Let us introduce, for the minimizer $q \in S_p[r]$, the following two objects

$$\mu := \lambda_m(q) \in \big(-\infty, (m\pi)^2\big), \qquad y(t) := E(t)/\sqrt{c_0}. \tag{3.7}$$

Then $y(t)$ is also an eigen-function associated with $\mu = \lambda_m(q)$

$$y'' + \mu y + q(t) y = 0. \tag{3.8}$$

As $q(t) \geqslant 0$, equality (3.5) can be rewritten as

$$q(t) = |y(t)|^{2/(p-1)} = |y(t)|^{2p^*-2}. \tag{3.9}$$

Substituting (3.9) into (3.8), $y(t)$ satisfies the following nonlinear Schrödinger equation

$$y'' + \mu y + \phi_{2p^*}(y) = 0, \tag{3.10}$$

because $q(t)y(t) = |y(t)|^{2p^*-2} y(t) = \phi_{2p^*}(y(t))$, where

$$\phi_\gamma(x) = |x|^{\gamma-2} x, \quad x \in \mathbb{R}, \ \gamma \in (1, \infty),$$

is the function in defining the γ-Laplacian. Note from (3.7) that μ and $y(t)$ are, respectively, the eigenvalue $\lambda_m(q)$ and an appropriate eigen-function for the minimal potential $q(t)$. Such a reduction has been worked out for the smallest periodic and Neumann eigenvalues in [22].

In the following, we consider $\mu \in (-\infty, (m\pi)^2)$ as a parameter. Let $y(t)$ be any non-zero solution of (3.10)-(1.2) such that $y(t)$ has exactly $(m+1)$ zeros in $[0,1]$. Define a potential $q(t)$ by (3.9). Then constraint (3.3) for $q(t)$ is transformed into the following condition on $y(t)$

$$\|y\|_{2p^*}^{2p^*/p} = r. \tag{3.11}$$

Eq. (3.10) is just the critical equation of problem (3.1), expressed in some eigen-function $y(t)$ with a parameter μ, because we have the following explanations to (3.10)–(3.11).

Lemma 3.1.

(i) *Suppose that* $q \in S_p[r]$ *is a minimizer of problem* (3.1)*. Then* (μ, y)*, defined by* (3.7)*, satisfies* (3.10)–(3.11)*.*
(ii) *Suppose that* $y(t)$ *is a solution of problem* (3.10)-(1.2) *with some parameter* $\mu \in (-\infty, (m\pi)^2)$ *so that* $y(t)$ *has exactly* $(m+1)$ *zeros in* $[0,1]$ *and satisfies condition* (3.11)*. Then* $q(t)$ *defined by* (3.9) *is a non-negative critical potential of problem* (3.1) *with the corresponding critical value* $\lambda_m(q) = \mu$*.*

Proof. We need only to notice that conclusion (ii) follows from the characterization in Lemma 2.1 on Dirichlet eigenvalues by simply rewriting (3.10) as (3.8). □

Note that the critical equation (3.10) is autonomous and is symmetric with respect to the transformation $y \to -y$. Finding those $y(t)$ as in Lemma 3.1(ii) can be transformed into periodic solutions of (3.10).

Lemma 3.2. *Let* $y(t)$*,* $t \in \mathbb{R}$*, be a solution of* (3.10) *such that* $y(t)$ *satisfies* (1.2) *and has exactly* $(m+1)$ *zeros in* $[0,1]$*. Then* $y(t)$ *satisfies*

$$y(t + 1/m) \equiv -y(t), \quad t \in \mathbb{R}, \tag{3.12}$$

and $y(t)$ *is a periodic solution of* (3.10) *of the minimal period* $2/m$*.*

Proof. Since $y(0) = 0$, the first integral of Eq. (3.10) is

$$\big(y'(t)\big)^2 + \mu\big(y(t)\big)^2 + \big|y(t)\big|^{2p^*}/p^* \equiv \big(y'(0)\big)^2. \tag{3.13}$$

From this, $y(t)$ is well defined for $t \in \mathbb{R}$. See Figs. 1 and 2 below.

Let

$$t_1 := \min\big\{t > 0: \ y(t) = 0\big\}$$

be the first positive zero of $y(t)$. From (3.13), we have $(y'(t_1))^2 = (y'(0))^2$ and $y'(t_1) = \pm y'(0)$. Since t_1 is the first positive zero, one has actually $y'(t_1) = -y'(0)$. Denote $\hat{y}(t) := -y(t+t_1)$, $t \in \mathbb{R}$. Then $\hat{y}(t)$ is also a solution of (3.10). As

$$\big(\hat{y}(0), \hat{y}'(0)\big) = \big(y(t_1), -y'(t_1)\big) = \big(0, y'(0)\big) = \big(y(0), y'(0)\big),$$

the uniqueness of solutions of initial value problems of Eq. (3.10) implies

$$y(t+t_1) \equiv -y(t), \quad t \in \mathbb{R}.$$

Hence the positive zeros of $y(t)$ must be

$$t_1 < 2t_1 < \cdots < nt_1 < \cdots.$$

Since $y(t)$ has precisely m zeros in $(0,1]$ and $t = 1$ is the mth positive zero, we have $t_1 = 1/m$. Thus $y(t)$ satisfies (3.12), which means that $y(t)$ is anti-periodic of the minimal period $1/m$. Consequently, $y(t)$ is periodic of the minimal period $2/m$. $\square$

For a T-periodic function $f \in L^p(\mathbb{R}/T\mathbb{Z})$, we write $\|f\|_{p,T} := \|f\|_{L^p[0,T]}$. Let $y(t)$ be as in Lemma 3.2. Then

$$\begin{aligned}
\|y\|_{2p^*}^{2p^*/p} &= \left(\int_0^1 |y(t)|^{2p^*}\,\mathrm{d}t\right)^{1/p} \\
&= \left(\frac{1}{2}\int_0^2 |y(t)|^{2p^*}\,\mathrm{d}t\right)^{1/p} \\
&= \left(\frac{m}{2}\int_0^{2/m} |y(t)|^{2p^*}\,\mathrm{d}t\right)^{1/p} \quad \big(\text{as } y(t) \text{ is } 2/m\text{-periodic}\big) \\
&= (m/2)^{1/p}\|y\|_{2p^*,2/m}^{2p^*/p}.
\end{aligned} \tag{3.14}$$

Lemma 3.3. *Let $\mu \in (-\infty, (m\pi)^2)$. Suppose that $y(t)$ is a periodic solution of* (3.10) *such that*

$$y(0) = 0, \qquad y(t) \text{ has the minimal period } 2/m, \qquad (m/2)^{1/p}\|y\|_{2p^*,2/m}^{2p^*/p} = r. \tag{3.15}$$

Then the potential $q(t)$ defined by (3.9) *has the following properties*

$$q(t) \text{ has the minimal period } 1/m, \qquad q \in S_p[r]. \tag{3.16}$$

Moreover, $q(t)$ is a (non-negative) critical potential of problem (3.1) *with the critical eigenvalue $\lambda_m(q) = \mu$.*

Proof. Properties (3.16) for $q(t)$ can be directly obtained from (3.15), with the help of formulas (3.11) and (3.14).

In order that $q(t)$ is a critical potential of problem (3.1), it suffices to have $y(1) = 0$. Since $y(0) = 0$ and $y(t)$ has the minimal period $2/m$, arguing as in the proof of Lemma 3.2, one can see that $y(t)$ satisfies (3.12). In particular, we have from (3.12)

$$y(1) = (-1)^m y(0) = 0,$$

proving the lemma. $\square$

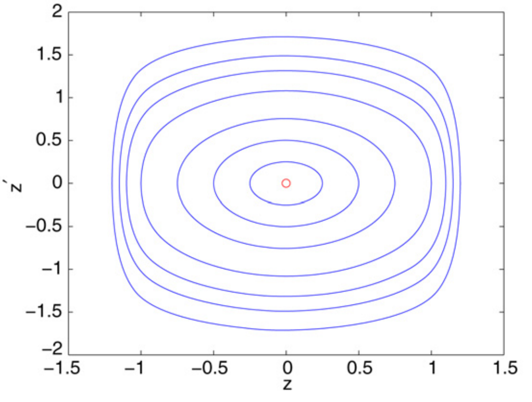

Fig. 1. Phase portrait of Eq. (3.18) with $p = 6/5$.

From these results, one sees that the minimal problems (3.1) can be completely determined by those parameters μ so that Eq. (3.10) has solutions which possess properties (3.15).

As a dynamical system, Eq. (3.10) has quite different phase portraits for the case $\mu > 0$ and $\mu < 0$. In order to use (3.10) and (3.15) to characterize minimal values $\mathbf{L}_{m,p}(r)$, we need to distinguish two cases.

3.2. Minimal values with small radius

By a small radius r we mean that $r \in (0, \mathbf{R}_{m,p})$. In this case, it follows from Lemma 2.8 that $\mu := \mathbf{L}_{m,p}(r) \in (0, (m\pi)^2)$. As in [22], let us scale Eq. (3.10) by

$$\nu = \sqrt{\mu} \in (0, m\pi), \qquad y(t) = \nu^{p-1} z(\nu t). \tag{3.17}$$

Then Eq. (3.10) is transformed into

$$z'' + z + \phi_{2p^*}(z) = 0. \tag{3.18}$$

The phase portrait of Eq. (3.18) is as in Fig. 1. It consists of an equilibrium $(0, 0)$ and a family of non-constant periodic solutions surrounding the equilibrium $(0, 0)$.

In order to study $\mathbf{L}_{m,p}(r)$, we need only to consider non-constant periodic solutions $z(t)$ of (3.18). Since Eq. (3.18) is autonomous and is symmetric with respect to the transformation $z \to -z$, after a translation of times t, solutions of (3.18) can be parameterized as $z(t; a)$ using a single parameter $a \in (0, \infty)$ by $\max_t z(t; a) = a$. The first integral for $z(t) = z(t; a)$ is

$$z'^2 + z^2 + |z|^{2p^*}/p^* = a^2 + a^{2p^*}/p^*. \tag{3.19}$$

Let us introduce a crucial parameter for $z(t; a)$ as

$$A = A_p(a) := a^{2p^*-2}/p^* > 0. \tag{3.20}$$

Then we have from (3.19)

$$z' = \frac{\mathrm{d}z}{\mathrm{d}t} = \pm\sqrt{a^2 + a^2 A - z^2 - |z|^{2p^*}/p^*}. \tag{3.21}$$

By the symmetry of solutions of (3.18) in the z–z' plane, the minimal period of $z(t;a)$ is

$$\mathbf{T}_p(a) = 4\int_0^a \frac{\mathrm{d}z}{\sqrt{a^2 + a^2 A - z^2 - z^{2p^*}/p^*}} = 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}} \quad \text{(by setting } z = ax\text{)}. \tag{3.22}$$

Define

$$\begin{aligned}\mathbf{U}_p(a) := \|z\|^{2p^*}_{2p^*,\mathbf{T}_p(a)} &= \int_0^{\mathbf{T}_p(a)} \left|z(t)\right|^{2p^*} \mathrm{d}t \\ &= 4\int_0^a \frac{z^{2p^*}\,\mathrm{d}z}{\sqrt{a^2 + a^2 A - z^2 - z^{2p^*}/p^*}} \quad \big(\text{by (3.21)}\big) \\ &= 4a^{2p^*}\int_0^1 \frac{x^{2p^*}\,\mathrm{d}x}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}} \\ &= 4(p^*)^p A^p \int_0^1 \frac{x^{2p^*}\,\mathrm{d}x}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}},\end{aligned} \tag{3.23}$$

because relation (3.20) implies $a^{2p^*} = (p^*A)^p$. Note that $\mathbf{T}_p(a)$ and $\mathbf{U}_p(a)$ can also be understood as functions of $A \in (0,\infty)$.

Lemma 3.4. *Given $p \in (1,\infty)$. The functions $\mathbf{T}_p(a)$ and $\mathbf{U}_p(a)$ possess the following properties.*

- *The period function $\mathbf{T}_p(a)$ is strictly decreasing in $a \in (0,\infty)$. Moreover,*

$$\mathbf{T}_p(0+) = 2\pi, \qquad \mathbf{T}_p(+\infty) = 0. \tag{3.24}$$

- *The function $\mathbf{U}_p(a)$ is strictly increasing in $a \in (0,\infty)$. Moreover,*

$$\mathbf{U}_p(0+) = 0, \qquad \mathbf{U}_p(+\infty) = +\infty. \tag{3.25}$$

Proof. By (3.20), $A = A_p(a)$ is strictly increasing in $a \in (0,\infty)$.

By formula (3.22), it is easy to see that $\mathbf{T}_p(a)$ is decreasing in $A \in (0,\infty)$ and therefore is decreasing in $a \in (0,\infty)$. Note that the integrand of (3.22) is controlled by

$$\frac{1}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}} \leqslant \frac{1}{\sqrt{1 - x^2}} \in \mathcal{L}^1.$$

When $a \downarrow 0$, one has $A = A_p(a) \downarrow 0$ and

$$\frac{1}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}} \to \frac{1}{\sqrt{1 - x^2}} \quad \text{for } x \in [0,1).$$

By the Lebesgue Dominated Convergence Theorem (LDCT), one has

$$\lim_{a\downarrow 0}\mathbf{T}_p(a)=4\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^2}}=2\pi.$$

When $a\uparrow\infty$, one has $A=A_p(a)\uparrow\infty$ and

$$\begin{aligned}\mathbf{T}_p(a)&=\frac{4}{\sqrt{A}}\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^{2p^*}+(1-x^2)/A}}\\&\leqslant\frac{4}{\sqrt{A}}\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^{2p^*}}}\to 0.\end{aligned}\tag{3.26}$$

We have the second limit of (3.24).

Function $\mathbf{U}_p(a)$ of (3.23) can be rewritten as

$$\mathbf{U}_p(a)=4(p^*)^p\int_0^1 f(x,A)\,\mathrm{d}x,\tag{3.27}$$

where

$$f(x,A)=\frac{A^p x^{2p^*}}{\sqrt{1-x^2+A(1-x^{2p^*})}}.$$

For any $x\in[0,1)$ fixed, we have

$$\frac{\partial f(x,A)}{\partial A}=\frac{A^{p-1}x^{2p^*}(2p(1-x^2)+(2p-1)A(1-x^{2p^*}))}{2(1-x^2+A(1-x^{2p^*}))^{3/2}}>0.$$

Hence the integrand of (3.23) is increasing in A. Now the monotonicity of $\mathbf{U}_p(a)$ follows simply from (3.27). The first limit of (3.25) can be obtained from (3.23)

$$\mathbf{U}_p(a)\leqslant 4a^{2p^*}\int_0^1\frac{x^{2p^*}}{\sqrt{1-x^2}}\,\mathrm{d}x\to 0\quad\text{as } a\downarrow 0.$$

For the second limit of (3.25), by formula (3.27) for $\mathbf{U}_p(a)$, we have

$$\mathbf{U}_p(a)=4(p^*)^p A^{p-1/2}\int_0^1\frac{x^{2p^*}}{\sqrt{1-x^{2p^*}+(1-x^2)/A}}\,\mathrm{d}x.$$

As $a\uparrow\infty$, by the LDCT, the integral above has the positive limit

$$\int_0^1\frac{x^{2p^*}}{\sqrt{1-x^{2p^*}}}\,\mathrm{d}x=:C_p.$$

Hence one has $\mathbf{U}_p(a)=(4(p^*)^p C_p+o(1))A^{p-1/2}\uparrow\infty$ as $a\uparrow\infty$. $\square$

Now we return back the minimal problem $\mathbf{L}_{m,p}(r)$. Let $\mu \in (0,(m\pi)^2)$. By the transformations (3.17), periodic solutions $y(t)$ of (3.10) of the minimal period $2/m$ are transformed to periodic solutions $z(t)$ of (3.18) of the minimal period $2\nu/m \in (0,2\pi)$ where $\nu = \sqrt{\mu}$.

Lemma 3.5. *Suppose that*

$$m \in \mathbb{N}, \qquad p \in (1,\infty), \qquad r \in (0,\mathbf{R}_{m,p}).$$

Then $\mathbf{L}_{m,p}(r) = \nu^2 \in (0,(m\pi)^2)$, *where* $\nu \in (0,m\pi)$ *satisfies*

$$\mathbf{T}_p(a) = 2\nu/m \quad \textit{and} \quad (m/2)^{1/p}\nu^{2-1/p}\big(\mathbf{U}_p(a)\big)^{1/p} = r \tag{3.28}$$

for some $a \in (0,\infty)$. *Moreover, system* (3.28) *for* (a,ν) *is equivalent to the following equation for* ν

$$(m/2)^{1/p}\nu^{2-1/p}\big(\mathbf{U}_p \circ \mathbf{T}_p^{-1}(2\nu/m)\big)^{1/p} = r. \tag{3.29}$$

Proof. Let $y(t)$ be the eigen-function associated with the minimal potential q. Then $y(t)$ can be transformed to a solution $z(t) = z(t;a)$ of (3.18) for some $a \in (0,\infty)$. Due to the requirement for minimal periods, one has $\mathbf{T}_p(a) = 2\nu/m$. Thus

$$y(t) = \nu^{p-1} z(\nu t; a).$$

Moreover, by (3.17),

$$\begin{aligned}
\|y\|_{2p^*,2/m}^{2p^*/p} &= \left(\int_0^{2/m} \big|y(t)\big|^{2p^*}\,\mathrm{d}t\right)^{1/p} \\
&= \left(\nu^{2p^*(p-1)} \int_0^{2/m} \big|z(\nu t;a)\big|^{2p^*}\,\mathrm{d}t\right)^{1/p} \\
&= \left(\nu^{2p-1} \int_0^{2\nu/m} \big|z(t;a)\big|^{2p^*}\,\mathrm{d}t\right)^{1/p} \\
&= \nu^{2-1/p}\big(\mathbf{U}_p(a)\big)^{1/p}.
\end{aligned}$$

By formula (3.14), the requirement (3.11) is the same as

$$r = (m/2)^{1/p}\|y\|_{2p^*,2/m}^{2p^*/p} = (m/2)^{1/p}\nu^{2-1/p}\big(\mathbf{U}_p(a)\big)^{1/p}.$$

That is, (ν, a) satisfies (3.28).

Due to the monotonicity of $\mathbf{T}_p(a)$, the first condition of (3.28) is

$$a = a_{m,p,r} = \mathbf{T}_p^{-1}(2\nu/m).$$

Substituting into the second condition, we know that $\nu = \sqrt{\mathbf{L}_{m,p}(r)}$ satisfies (3.29). $\square$

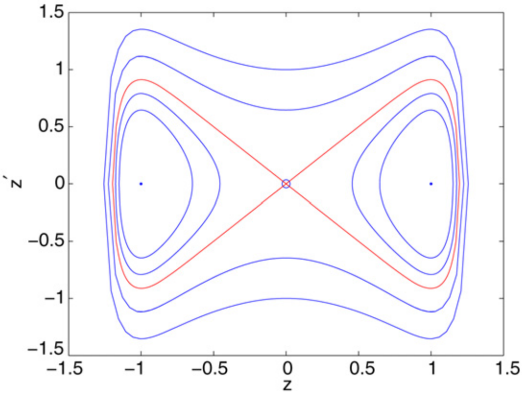

Fig. 2. Phase portrait of Eq. (3.30) with $p = 6/5$.

3.3. *Minimal values with large radius*

By a large radius r we mean that $r \in (\mathbf{R}_{m,p}, \infty)$. In this case, we have $\mathbf{L}_{m,p}(r) \in (-\infty, 0)$. Let us normalize Eq. (3.10) by

$$\nu = \sqrt{-\mu} \in (0, \infty), \qquad y(t) = \nu^{p-1} z(\nu t).$$

Then Eq. (3.10) is transformed into

$$z'' - z + \phi_{2p^*}(z) = 0. \tag{3.30}$$

The phase portrait of Eq. (3.30) is as in Fig. 2.

We have met the normalized equation (3.30) in [22] in the study of the minimal values of smallest periodic eigenvalues of Hill's operators, where only positive periodic solutions of (3.30) are needed. However, in the present case, by Lemma 3.3, we need to consider sign-changing periodic solutions of (3.30), which can be parameterized as $z(t; a)$ using $\max_t z(t; a) = a$. However, different from the preceding case, the parameter a takes values in

$$a \in (\mathbf{b}_p, \infty), \quad \mathbf{b}_p := (p^*)^{(p-1)/2}.$$

For the obtention of such a parameter $\mathbf{b}_p$, see [22]. From the basic limit $\lim_{x \downarrow 0} x^x = 1$, we have

$$\lim_{p \downarrow 1} \mathbf{b}_p = 1. \tag{3.31}$$

Moreover, one has

$$\inf_{p \in (1,\infty)} \mathbf{b}_p = 1, \qquad \sup_{p \in (1,\infty)} \mathbf{b}_p = \sqrt{e}.$$

The minimal period of $z(t; a)$ is now

$$\hat{\mathbf{T}}_p(a) = 4 \int_0^a \frac{\mathrm{d}z}{\sqrt{z^2 - a^2 + (a^{2p^*} - z^{2p^*})/p^*}}$$

$$= 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2 - 1 + A(1 - x^{2p^*})}}. \tag{3.32}$$

Here $A = A_p(a)$ is as in (3.20). However, as $a > \mathbf{b}_p$, $A = A_p(a)$ is now in $(1, \infty)$. Similarly, we introduce

$$\begin{aligned}\hat{\mathbf{U}}_p(a) &:= \int_0^{\hat{\mathbf{T}}_p(a)} \left|z(t)\right|^{2p^*} \mathrm{d}t \\ &= 4\int_0^a \frac{z^{2p^*}}{\sqrt{z^2 - a^2 + (a^{2p^*} - z^{2p^*})/p^*}} \mathrm{d}z \\ &= 4(p^*A)^p \int_0^1 \frac{x^{2p^*}}{\sqrt{x^2 - 1 + A(1 - x^{2p^*})}} \mathrm{d}x. \end{aligned} \tag{3.33}$$

The following properties of $\hat{\mathbf{T}}_p(a)$ and $\hat{\mathbf{U}}_p(a)$ of (3.32) and (3.33) can be verified as before.

Lemma 3.6.

- *The period function $\hat{\mathbf{T}}_p(a)$ is strictly decreasing in $a \in (\mathbf{b}_p, \infty)$. Moreover, $\hat{\mathbf{T}}_p(\mathbf{b}_p+) = +\infty$ and $\hat{\mathbf{T}}_p(+\infty) = 0$.*
- *The function $\hat{\mathbf{U}}_p(a)$ is strictly increasing in $a \in (\mathbf{b}_p, \infty)$. Moreover, $\hat{\mathbf{U}}_p(+\infty) = +\infty$ and*

$$\hat{\mathbf{U}}_p(\mathbf{b}_p+) = 4(p^*)^p \int_0^1 \frac{x^{2p^*}}{\sqrt{x^2 - x^{2p^*}}} \mathrm{d}x = \frac{2p^p \mathbf{B}(p, 1/2)}{(p-1)^{p-1}}. \tag{3.34}$$

Here $\mathbf{B}(\cdot,\cdot)$ is the Beta function of Euler.

Proof. We need only to verify (3.34). Let $a \downarrow \mathbf{b}_p$ in (3.33). We have $A = A_p(a) \downarrow 1$. The limit is then given by (3.34). □

Arguing as in the proof of Lemma 3.5, we can use $\hat{\mathbf{T}}_p(a)$ and $\hat{\mathbf{U}}_p(a)$ of (3.32) and (3.33) to characterize $\nu = \sqrt{-\mathbf{L}_{m,p}(r)} \in (0, \infty)$.

Lemma 3.7. *Suppose that*

$$m \in \mathbb{N}, \qquad p \in (1, \infty), \qquad r \in (\mathbf{R}_{m,p}, \infty).$$

Then $\mathbf{L}_{m,p}(r) = -\nu^2 \in (-\infty, 0)$, where $\nu \in (0, \infty)$ satisfies

$$\hat{\mathbf{T}}_p(a) = 2\nu/m \quad \textit{and} \quad (m/2)^{1/p} \nu^{2-1/p} \big(\hat{\mathbf{U}}_p(a)\big)^{1/p} = r$$

for some $a \in (\mathbf{b}_p, \infty)$. This is equivalent to the following equation for $\nu \in (0, \infty)$

$$(m/2)^{1/p} \nu^{2-1/p} \big(\hat{\mathbf{U}}_p \circ \hat{\mathbf{T}}_p^{-1}(2\nu/m)\big)^{1/p} = r. \tag{3.35}$$

In summary, let $p \in (1, \infty)$ be given. By Lemmas 3.4 and 3.6, the inverse $\mathbf{T}_p^{-1}(x)$ is well defined for $x \in (0, 2\pi)$, and the inverse $\hat{\mathbf{T}}_p^{-1}(x)$ is well defined for $x \in (0, \infty)$. Now we can introduce a function $\mathbf{Z}_{1,p} : (-\infty, \pi^2] \to [0, \infty)$ by

$$\mathbf{Z}_{1,p}(x) = \begin{cases} 0 & \text{for } x = \pi^2, \\ \frac{1}{4}(2\sqrt{x})^{2-1/p}(\mathbf{U}_p(\mathbf{T}_p^{-1}(2\sqrt{x})))^{1/p} & \text{for } x \in (0, \pi^2), \\ \mathbf{R}_{1,p} = \mathbf{K}(2p^*) & \text{for } x = 0, \\ \frac{1}{4}(2\sqrt{-x})^{2-1/p}(\hat{\mathbf{U}}_p(\hat{\mathbf{T}}_p^{-1}(2\sqrt{-x})))^{1/p} & \text{for } x \in (-\infty, 0). \end{cases} \tag{3.36}$$

The function $\mathbf{Z}_{1,p}$ is well defined and is continuous, following from Lemmas 3.4 and 3.6.

Theorem 3.8. *Let $p \in (1, \infty)$ be given.*

- *The function $\mathbf{Z}_{1,p} : (-\infty, \pi^2] \to [0, \infty)$ of (3.36) is a decreasing homeomorphism.*
- *Let $m \in \mathbb{N}$. The minimal values $\mathbf{L}_{m,p}(r)$ are given by*

$$\mathbf{L}_{m,p}(r) = m^2 \mathbf{Z}_{1,p}^{-1}\big(r/m^2\big), \quad r \in [0, \infty). \tag{3.37}$$

- *In particular, we have*

$$\mathbf{L}_{1,p}(r) = \mathbf{Z}_{1,p}^{-1}(r), \quad r \in [0, \infty). \tag{3.38}$$

Proof. Using the function $\mathbf{Z}_{1,p}$ of (3.36), equality (3.29) for the case $r \in (0, \mathbf{R}_{m,p})$ and equality (3.35) for the case $r \in (\mathbf{R}_{m,p}, \infty)$ can be rewritten in the unified form

$$\mathbf{Z}_{1,p}\big(\mathbf{L}_{m,p}(r)/m^2\big) = r/m^2. \tag{3.39}$$

This is also true for the special radii $r = 0$ and $r = \mathbf{R}_{m,p}$. By Lemma 2.7, we have known that $\mathbf{L}_{m,p}(r)$ is a decreasing homeomorphism of $r \in [0, \infty)$. From (3.39), it is necessary that $\mathbf{Z}_{1,p}(x)$ is also a decreasing homeomorphism mapping $(-\infty, \pi^2]$ onto $[0, \infty)$. Due to this, (3.39) can be rewritten as (3.37). $\square$

Proof of Theorem 1.3. For the case $p \in (1, \infty)$, amplification relation (1.7) follows simply from (3.37) and (3.38). For the case $p = \infty$, (1.7) is trivial by equality (2.1). For the case $p = 1$, by letting $p \downarrow 1$ in (1.7), we get (1.7) for $p = 1$ because we have the limiting equalities (2.2) for $\mathbf{L}_{m,1}(r)$. $\square$

We remark that the zeros $r = \mathbf{R}_{m,p}$ of $\mathbf{L}_{m,p}(r)$ in (2.8) satisfy $\mathbf{R}_{m,p} = m^2 \mathbf{R}_{1,p}$. This is consistent with amplification relation (1.7).

4. Limiting approach to minimal values $\mathbf{L}_{1,1}(r)$

In this section, we will complete the proof of Theorem 1.4, using the limiting approach as in [22]. We need to distinguish two cases for different radius r.

4.1. Small radius

In this subsection we always assume that $r \in (0, \mathbf{R}_{1,1}) = (0, 4)$.

Since $\mathbf{K}(\gamma)$ is strictly decreasing in $\gamma \in [1, \infty]$, it follows from formula (2.8) that

$$0 < r < \mathbf{R}_{1,1} < \mathbf{R}_{1,p} = \mathbf{K}(2p^*) \quad \text{for } p \in (1, \infty).$$

Let $p \in (1, \infty)$. By Lemma 3.5, $\nu_p = \nu_{p,r} := \sqrt{\mathbf{L}_{1,p}(r)} \in (0, \pi)$ is determined by system (3.28) where $m = 1$. Let us introduce

$$a_p = a_{p,r} := \mathbf{T}_p^{-1}(2\nu_p), \qquad A_p = A_{p,r} := a_p^{2p^*-2}/p^*. \tag{4.1}$$

System (3.28) is

$$\mathbf{T}_p(a_p) = 2\nu_p \quad \text{and} \quad \nu_p^{2-1/p}\big(\mathbf{U}_p(a_p)/2\big)^{1/p} = r. \tag{4.2}$$

In the following, we consider a_p and A_p of (4.1) as functions of $p \in (1, \infty)$. Some crucial observations are as follows.

Lemma 4.1. *There holds the following limit*

$$\lim_{p \downarrow 1} A_p = \cot^2(\nu_1/2) \in (0, \infty), \tag{4.3}$$

where $\nu_1 = \nu_{1,r} := \sqrt{\mathbf{L}_{1,1}(r)} \in (0, \pi)$. *In particular,*

$$\lim_{p \downarrow 1} a_p = 1. \tag{4.4}$$

Proof. Recall from (3.20) that $a_p = \mathbf{b}_p A_p^{(p-1)/2}$. One sees that (4.4) is a simple consequence of (4.3) because we have the limit (3.31). Hence we need only to prove (4.3).

By (2.2), we have known

$$\lim_{p \downarrow 1} \nu_p = \nu_1 \in (0, \pi). \tag{4.5}$$

- We assert that

$$\liminf_{p \downarrow 1} A_p > 0.$$

Otherwise, there exists $p_n \downarrow 1$ such that $\lim_{n \to \infty} A_{p_n} = 0$. In (3.22), we have

$$\frac{1}{\sqrt{1 - x^2 + A_{p_n}(1 - x^{2p_n^*})}} \leqslant \frac{1}{\sqrt{1 - x^2}} \in \mathcal{L}^1,$$

and, when $n \to \infty$,

$$\frac{1}{\sqrt{1 - x^2 + A_{p_n}(1 - x^{2p_n^*})}} \to \frac{1}{\sqrt{1 - x^2}}$$

for all $x \in [0, 1)$. Now the LDCT shows that

$$2\nu_{p_n} \equiv \mathbf{T}_{p_n}(a_n) \to 4\int_0^1 \frac{dx}{\sqrt{1 - x^2}} = 2\pi.$$

This is a contradiction, because (4.5) implies that $2\nu_{p_n} \to 2\nu_1 < 2\pi$.

- We assert that

$$\limsup_{p\downarrow 1} A_p < \infty.$$

Otherwise, there exists $p_n \downarrow 1$ such that $\lim_{n\to\infty} A_{p_n} = +\infty$. By (3.26), we have

$$2\nu_{p_n} \equiv \mathbf{T}_{p_n}(a_{p_n}) \leqslant \frac{4}{\sqrt{A_{p_n}}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^{2p_n^*}}} \to 0.$$

Again, this is a contradiction with (4.5).

For any sequence p_n so that $p_n \downarrow 1$, we know from the two assertions above that there exists some subsequence of $\{p_n\}$, still denoted by $\{p_n\}$, such that

$$\lim_{n\to\infty} A_{p_n} = \alpha_0 \in (0, \infty). \tag{4.6}$$

Let now $p = p_n$ and $a = a_{p_n}$ in (3.22). Due to the limit (4.6), by applying the LDCT to (3.22), we obtain

$$\begin{aligned}
2\nu_1 &= \lim_{n\to\infty} \mathbf{T}_{p_n}(a_{p_n}) \\
&= \lim_{n\to\infty} 4 \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2 + A_{p_n}(1 - x^{2p_n^*})}} \\
&= 4 \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2 + \alpha_0}} \\
&= 4\cot^{-1}\sqrt{\alpha_0}.
\end{aligned}$$

Thus $\alpha_0 = \cot^2(\nu_1/2)$. As the limit α_0 of A_{p_n} is independent of the choice of sequences p_n, the limit (4.3) does exist and is equal to $\cot^2(\nu_1/2)$. $\square$

Next we can compute the limit of $\mathbf{U}_p(a_p)$ as $p \downarrow 1$.

Lemma 4.2. *One has the following limit*

$$\lim_{p\downarrow 1} \mathbf{U}_p(a_p) = 4\cot(\nu_1/2) \in (0, \infty). \tag{4.7}$$

Proof. In this case, $\mathbf{U}_p(a)$ is given by (3.23). However, formula (3.23) is not convenient in finding the limit. As in [22], let us compute $\mathbf{U}_p(a)$ in another way. Multiplying Eq. (3.18) by $z(t)$ and integrating over one period $[0, \mathbf{T}_p(a)]$, we get

$$\begin{aligned}
\mathbf{U}_p(a) &= \int_0^{\mathbf{T}_p(a)} \left|z(t)\right|^{2p^*} \mathrm{d}t \\
&= \int_0^{\mathbf{T}_p(a)} \left(z'^2(t) - z^2(t)\right) \mathrm{d}t
\end{aligned}$$

$$= \int_0^{\mathbf{T}_p(a)} \left(a^2 + a^{2p^*}/p^* - 2z^2(t) - \left|z(t)\right|^{2p^*}/p^*\right) \mathrm{d}t \quad \big(\text{by (3.19)}\big)$$

$$= 4 \int_0^a \frac{a^2 + a^{2p^*}/p^* - 2z^2 - z^{2p^*}/p^*}{\sqrt{a^2 + a^{2p^*}/p^* - (z^2 + z^{2p^*}/p^*)}} \mathrm{d}z \quad \big(\text{by (3.21)}\big)$$

$$= 4a^2 \int_0^1 \frac{1 - 2x^2 + A(1 - x^{2p^*})}{\sqrt{1 - x^2 + A(1 - x^{2p^*})}} \mathrm{d}x.$$

At this moment, we have the convergence results (4.3) and (4.4). Now we can apply the LDCT to obtain

$$\lim_{p\downarrow 1} \mathbf{U}_p(a_p) = \lim_{p\downarrow 1} 4a_p^2 \int_0^1 \frac{1 - 2x^2 + A_p(1 - x^{2p^*})}{\sqrt{1 - x^2 + A_p(1 - x^{2p^*})}} \mathrm{d}x$$

$$= 4 \int_0^1 \frac{1 - 2x^2 + \cot^2(\nu_1/2)}{\sqrt{1 - x^2 + \cot^2(\nu_1/2)}} \mathrm{d}x$$

$$= 4 \cot(\nu_1/2).$$

This proves (4.7). $\square$

Lemma 4.3. *The value $\nu_1 = \nu_{1,r} := \sqrt{\mathbf{L}_{1,1}(r)} \in (0, \pi)$ is determined by*

$$2\nu_1 \cot(\nu_1/2) = r. \tag{4.8}$$

Proof. Based on limits (4.5) and (4.7), by letting $p \downarrow 1$ in the second equality of (4.2), we get

$$r = \nu_p^{2-1/p} \big(\mathbf{U}_p(a_p)/2\big)^{1/p} \to \nu_1 \cdot 2\cot(\nu_1/2).$$

This gives Eq. (4.8) for ν_1. $\square$

4.2. Large radius

In this subsection we always assume that $r > \mathbf{R}_{1,1} = 4$.

As $\mathbf{R}_{1,p} = \mathbf{K}(2p^*) \downarrow 4$ when $p \downarrow 1$, there exists some $p_r > 1$ such that

$$r > \mathbf{R}_{1,p} \quad \text{for all } p \in (1, p_r].$$

In the following we only consider $p \in (1, p_r]$. By Lemma 3.7, $\mathbf{L}_{1,p}(r) < 0$ in this case. Denote

$$\nu_p = \nu_{p,r} := \sqrt{-\mathbf{L}_{1,p}(r)} > 0,$$

$$a_p = a_{p,r} := \hat{\mathbf{T}}_p^{-1}(2\nu_{p,r}) > \mathbf{b}_p = (p^*)^{(p-1)/2},$$

$$A_p = A_{p,r} := (a_{p,r})^{2p^*-2}/p^* > 1.$$

Then ν_p and a_p satisfy

$$\hat{\mathbf{T}}_p(a_p) = 2\nu_p \quad \text{and} \quad \nu_p^{2-1/p}\big(\hat{\mathbf{U}}_p(a_p)/2\big)^{1/p} = r. \tag{4.9}$$

Note that

$$\lim_{p\downarrow 1} \nu_p = \nu_1 = \nu_{1,r} := \sqrt{-\mathbf{L}_{1,1}(r)} \in (0,\infty).$$

Lemma 4.4. *One has the following limits*

$$\lim_{p\downarrow 1} A_p = \coth^2(\nu_1/2) \in (1,\infty), \tag{4.10}$$

$$\lim_{p\downarrow 1} a_p = 1. \tag{4.11}$$

Proof. • We assert that

$$\liminf_{p\downarrow 1} A_p \geqslant 1 + 4\exp(-\nu_1) > 1. \tag{4.12}$$

By (3.32), we have

$$\begin{aligned}
\hat{\mathbf{T}}_p(a_p) &= 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{A_p - 1 + x^2 - A_p x^{2p^*}}} \\
&> 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{A_p - 1 + x^2}} \\
&= 4\log\frac{1 + A_p^{1/2}}{\sqrt{A_p - 1}} \\
&> 4\log\frac{2}{\sqrt{A_p - 1}}.
\end{aligned}$$

An elementary computation shows that

$$A_p > 1 + 4\exp\big(-\hat{\mathbf{T}}_p(a_p)/2\big) = 1 + 4\exp(-\nu_p),$$

proving (4.12).

• We assert that

$$\limsup_{p\downarrow 1} A_p \leqslant 1 + (\pi/\nu_1)^2 < +\infty. \tag{4.13}$$

Note that

$$\frac{1 - x^{2p^*}}{1 - x^2} \geqslant 1, \quad x \in [0,1).$$

Thus

$$\hat{\mathbf{T}}_p(a_p) = 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{A_p\frac{1-x^{2p^*}}{1-x^2}-1}}$$
$$< 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{A_p-1}}$$
$$= \frac{2\pi}{\sqrt{A_p-1}}.$$

We have

$$A_p < 1 + \left(2\pi/\hat{\mathbf{T}}_p(a_p)\right)^2 = 1 + (2\pi/\nu_p)^2,$$

proving (4.13).

By (4.12) and (4.13), for any sequence $p_n \in (1, p_r]$ with $p_n \downarrow 1$, one can choose some subsequence, still denoted by p_n, such that A_{p_n} has some limit $\alpha_0 \in (1, \infty)$. Applying the LDCT to

$$2\nu_{p_n} \equiv \mathbf{T}_{p_n}(a_{p_n}) = 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2-1+A_{p_n}(1-x^{2p_n^*})}},$$

we can obtain

$$2\nu_1 = \lim_{n\to\infty} 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2-1+A_{p_n}(1-x^{2p_n^*})}}$$
$$= 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2-1+\alpha_0}}$$
$$= 4\coth^{-1}\sqrt{\alpha_0}.$$

That is, $\alpha_0 = \coth^2(\nu_1/2)$, which is independent of $\{p_n\}$. We have proved (4.10). Now (4.11) is a simple consequence of (4.10), like the observation in Lemma 4.1. □

Lemma 4.5. *One has*

$$\lim_{p\downarrow 1} \hat{\mathbf{U}}_p(a_p) = 4\coth(\nu_1/2). \tag{4.14}$$

Proof. As did in the proof of Lemma 4.2, we rewrite (3.33) in another form by exploiting Eq. (3.30)

$$\hat{\mathbf{U}}_p(a) = 4\int_0^a \frac{2z^2-a^2+(a^{2p^*}-z^{2p^*})/p^*}{\sqrt{z^2-a^2+(a^{2p^*}-z^{2p^*})/p^*}}\,\mathrm{d}z$$
$$= 4a^2\int_0^1 \frac{2x^2-1+A(1-x^{2p^*})}{\sqrt{x^2-1+A(1-x^{2p^*})}}\,\mathrm{d}x,$$

where $A = A_p(a) > 1$ is as in (3.20).

Using the facts (4.10) and (4.11), we can apply the LDCT to obtain

$$\begin{aligned}\lim_{p\downarrow 1}\hat{\mathbf{U}}_p(a_p) &= 4\int_0^1 \frac{2x^2-1+\coth^2(\nu_1/2)}{\sqrt{x^2-1+\coth^2(\nu_1/2)}}\,\mathrm{d}x\\ &= 4\coth(\nu_1/2),\end{aligned}$$

proving (4.14). □

Lemma 4.6. *The value $\nu_1=\sqrt{-\mathbf{L}_{1,1}(r)}>0$ satisfies the following equation*

$$2\nu_1\coth(\nu_1/2)=r. \tag{4.15}$$

Proof. The second equation of (4.9) is

$$\nu_p^{2-1/p}\big(\hat{\mathbf{U}}_p(a_p)/2\big)^{1/p}=r,\quad p\in(1,p_r].$$

By letting $p\downarrow 1$, we can use (4.14) to obtain (4.15). □

Proof of Theorem 1.4. We have known that $\nu_1:=\sqrt{\mathbf{L}_{1,1}(r)}$ is determined by (4.8) for $r\in(0,4)$, and $\nu_1:=\sqrt{-\mathbf{L}_{1,1}(r)}$ is determined by (4.15) for $r\in(4,\infty)$. Moreover, $\mathbf{L}_{1,1}(0)=\pi^2$ and $\mathbf{L}_{1,1}(4)=0$. Using the function $\mathbf{Z}_1(x)$ in (1.8), $\mathbf{L}_{1,1}(r)$ can be written as (1.9) in a unified way. □

Remark 4.7. For any $r>0$, $\mathbf{L}_{1,1}(r)$ and $\mathbf{L}_{m,1}(r)$ cannot be realized by any potential in $B_1[r]$. In fact, one can show that as $p\downarrow 1$, the minimizers $q_{p,r}(t)\in B_p[r]$ of $\mathbf{L}_{1,p}(r)$ will tend in some weak sense to the Dirac measure $r\delta_{1/2}(t)$ which is outside the L^1 space. This phenomenon has some close connection with the theory for generalized ordinary differential equations [12].

4.3. Asymptotical formulas of $\mathbf{L}_{m,p}(r)$ in large radius

Note that the function $\mathbf{Z}_1(x)$ of (1.8) has the asymptotical expression

$$\mathbf{Z}_1(x)=2\sqrt{-x}\coth(\sqrt{-x}/2)=2\sqrt{-x}+O\big(\sqrt{-x}\exp(-\sqrt{x})\big)\quad\text{for } x\ll -1.$$

Thus

$$-r^2/4-o(1)=\mathbf{L}_{1,1}(r)<-r^2/4\quad\text{for all } r\gg 1.$$

By (1.7), one has, for $m\in\mathbb{N}$,

$$\mathbf{L}_{m,1}(r)=-r^2/\big(4m^2\big)-o(1)\quad\text{for all } r\gg 1. \tag{4.16}$$

For the case $p=\infty$, $\mathbf{L}_{m,\infty}(r)$ is given by (2.1). The asymptotical behavior of $\mathbf{L}_{m,\infty}(r)$ in $r\gg 1$ is quite clear. For $m\in\mathbb{N}$ and $p\in(1,\infty)$, the orders of the minimal functions $\mathbf{L}_{m,p}(r)$ in r large can be found explicitly.

Corollary 4.8. *Define, for $p \in (1, \infty)$,*

$$\mathbf{C}(p) := \left(\frac{(p-1)^{p-1}}{p^p \mathbf{B}(p, 1/2)} \right)^{2/(2p-1)}.$$

Then, for any $m \in \mathbb{N}$, one has the following order of $\mathbf{L}_{m,p}(r)$ in $r \gg 1$

$$\lim_{r\uparrow\infty} \frac{-\mathbf{L}_{m,p}(r)}{r^{(2p)^*}} = m^{2(1-(2p)^*)}\mathbf{C}(p) = m^{-2/(2p-1)}\mathbf{C}(p). \tag{4.17}$$

Proof. By (1.7), we need only to prove (4.17) for $m = 1$. Let $p \in (1, \infty)$ be fixed. For $r > \mathbf{R}_{1,p}$, we know that $\nu_r := \sqrt{-\mathbf{L}_{1,p}(r)}$ and $a_r := \hat{\mathbf{T}}_p^{-1}(2\nu_r)$ are determined by

$$\hat{\mathbf{T}}_p(a_r) = 2\nu_r \quad \text{and} \quad \nu_r^{2-1/p}\big(\hat{\mathbf{U}}_p(a_r)/2\big)^{1/p} = r.$$

Here $\hat{\mathbf{T}}_p(a)$ and $\hat{\mathbf{U}}_p(a)$ are given by (3.32) and (3.33), respectively. The second equality implies that

$$-\mathbf{L}_{1,p}(r)/r^{(2p)^*} = \nu_r^2/r^{(2p)^*} = \big(2/\hat{\mathbf{U}}_p(a_r)\big)^{2/(2p-1)}. \tag{4.18}$$

As $\lim_{r\uparrow\infty} \nu_r = \infty$, we know that

$$\lim_{r\uparrow\infty} a_r = \lim_{r\uparrow\infty} \hat{\mathbf{T}}_p^{-1}(2\nu_r) = \mathbf{b}_p = (p^*)^{(p-1)/2}.$$

See Lemma 3.6. Now (3.34) implies

$$\lim_{r\uparrow\infty} \hat{\mathbf{U}}_p(a_r)/2 = p^p \mathbf{B}(p, 1/2)/(p-1)^{p-1}.$$

By letting $r \uparrow \infty$ in (4.18), we can get (4.17) for the case $m = 1$. $\square$

Remark 4.9. The orders (4.17) of $\mathbf{L}_{1,p}(r)$ are consistent with (4.16) for $p = 1$ and formula (2.1) for $p = \infty$, because we have $\lim_{p\downarrow 1} \mathbf{C}(p) = 1/4$ and $\lim_{p\uparrow\infty} \mathbf{C}(p) = 1$. Note that, as p increases from 1 to ∞, the order $(2p)^*$ of $-\mathbf{L}_{1,p}(r)$ in $r \gg 1$ decreases from 2 to 1. One may compare these results with the minimal values of the smallest periodic and the smallest Neumann eigenvalues in [22, Section 6].

5. Maximal values $\mathbf{M}_{m,p}(r)$ and $\mathbf{M}_{m,1}(r)$

5.1. Variational approach to $\mathbf{M}_{m,p}(r)$

Let (m, p, r) be as in (1.4). By (2.1) and Lemma 2.3, we have a trivial lower bound for maximal values $\mathbf{M}_{m,p}(r)$

$$\mathbf{M}_{m,p}(r) \geqslant (m\pi)^2 + r. \tag{5.1}$$

In case $p = \infty$, (5.1) is an equality.

In the following we assume that (m, p, r) is as in (2.4). As in Section 3, $\mu := \mathbf{M}_{m,p}(r)$ can be characterized using variational method. In this case, as the maximizers $q \in S_p[r]$ are non-positive, see Lemma 2.5, the expression (3.9) for q using y should be replaced by

$$q(t) = -\big|y(t)\big|^{2/(p-1)} = -\big|y(t)\big|^{2p^*-2}.$$

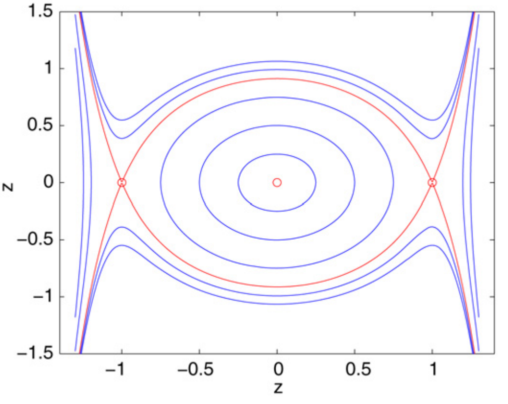

Fig. 3. Phase portrait of Eq. (5.5) where $p = 6/5$.

Now the critical equation (3.10) is changed to

$$y'' + \mu y - \phi_{2p^*}(y) = 0. \tag{5.2}$$

This is the critical equation for $\mathbf{M}_{m,p}(r)$. As $y(t)$ is an eigen-function satisfying (1.2), Eq. (5.2) shows that $y(t)$ cannot be constant. Consequently, $q(t) = -|y(t)|^{2p^*-2}$ is non-constant. Thus (5.1) is actually strict

$$\mathbf{M}_{m,p}(r) > (m\pi)^2 + r, \quad m \in \mathbb{N},\ p \in [1, \infty),\ r > 0. \tag{5.3}$$

Now we consider μ in the critical equation (5.2) as a parameter, taking values in $((m\pi)^2 + r, \infty)$. The conditions for μ to be $\mathbf{M}_{m,p}(r)$ are as follows. As $y(t)$ is an eigen-function associated with the mth Dirichlet eigenvalue, we shall seek solutions $y(t)$ of (5.2) so that $y(t)$ has precisely $(m + 1)$ zeros in $[0, 1]$, including 0 and 1. Arguing as in Section 3, we need to consider sign-changing periodic solutions $y(t)$ of (5.2) which have the minimal period $2/m$ and fulfill the requirement (3.11).

Let us normalize Eq. (5.2) by

$$\mu = \nu^2, \quad \nu > \sqrt{(m\pi)^2 + r}, \qquad y(t) = \nu^{p-1} z(\nu t). \tag{5.4}$$

Then Eq. (5.2) is transformed to

$$z'' + z - \phi_{2p^*}(z) = 0. \tag{5.5}$$

The phase portrait of Eq. (5.5) is as in Fig. 3. Notice that Eq. (5.5) is a nonlinear autonomous Schrödinger equation, which has three equilibria: $z = 0$ which is elliptic, and $z = \pm 1$ which are hyperbolic. Eq. (5.5) has a family of sign-changing periodic solutions surrounding the equilibrium $(0, 0)$.

Due to the autonomy and symmetry of Eq. (5.5), sign-changing periodic solutions can be parameterized as $z(t; a)$ by $\max_t z(t; a) = a$. Here the parameter a takes value in $(0, 1)$. Integrating (5.5), we know that $z(t) = z(t; a)$ satisfies

$$z'^2 + z^2 - |z|^{2p^*}/p^* = a^2 - a^{2p^*}/p^*.$$

The minimal period $z(t;a)$ is given by

$$\check{\mathbf{T}}_p(a) = 4\int_0^a \frac{\mathrm{d}z}{\sqrt{a^2 - z^2 - (a^{2p^*} - z^{2p^*})/p^*}} = 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^2 - B(1 - x^{2p^*})/p^*}}. \tag{5.6}$$

Here $B = B_p(a)$ is defined by

$$B = B_p(a) := a^{2p^*-2} = a^{2/(p-1)} \in (0,1). \tag{5.7}$$

Let us introduce

$$\begin{aligned}\check{\mathbf{U}}_p(a) = \left\|z(\cdot;a)\right\|_{2p^*,\check{\mathbf{T}}_p(a)}^{2p^*} &= \int_0^{\check{\mathbf{T}}_p(a)} \left|z(t;a)\right|^{2p^*} \mathrm{d}t \\ &= 4\int_0^a \frac{z^{2p^*}}{\sqrt{a^2 - a^{2p^*}/p^* - (z^2 - z^{2p^*}/p^*)}}\,\mathrm{d}z \\ &= 4a^{2p^*}\int_0^1 \frac{x^{2p^*}}{\sqrt{1 - x^2 - B(1 - x^{2p^*})/p^*}}\,\mathrm{d}x \\ &= 4B^p\int_0^1 \frac{x^{2p^*}}{\sqrt{1 - x^2 - B(1 - x^{2p^*})/p^*}}\,\mathrm{d}x. \end{aligned}\tag{5.8}$$

Formulas (5.6) and (5.8) show that $\check{\mathbf{T}}_p(a)$ and $\check{\mathbf{U}}_p(a)$ can be considered as functions of $B = B_p(a) \in (0,1)$ defined by (5.7). Note that $\check{\mathbf{T}}_p(a)$ is strictly increasing in $a \in (0,1)$. Moreover,

$$\check{\mathbf{T}}_p(0+) = 2\pi \quad \text{and} \quad \check{\mathbf{T}}_p(1-) = +\infty.$$

In order to study

$$\nu_p = \nu_{m,p,r} := \sqrt{\mathbf{M}_{m,p}(r)} > \sqrt{(m\pi)^2 + r}, \tag{5.9}$$

we need to consider periodic solutions $z(t;a)$ of (5.5) which have the minimal period $2\nu_p/m$. Using the functions $\check{\mathbf{T}}_p(a)$ and $\check{\mathbf{U}}_p(a)$, computation as in preceding sections can yield the following characterization on $\mathbf{M}_{m,p}(r)$.

Theorem 5.1. *Given $m \in \mathbb{N}$, $p \in (1,\infty)$ and $r > 0$. Let $\nu_p = \nu_{m,p,r}$ be as in* (5.9)*.*

(i) *There exists the unique $a_p = a_{m,p,r} \in (0,1)$ such that*

$$\check{\mathbf{T}}_p(a_p) = 2\nu_p/m \quad \textit{and} \quad (m/2)^{1/p}\nu_p^{2-1/p}\big(\check{\mathbf{U}}_p(a_p)\big)^{1/p} = r.$$

(ii) *The maximal value* $\mathbf{M}_{m,p}(r)$ *is determined by*

$$\check{\mathbf{Y}}_{1,p}\big(\mathbf{M}_{m,p}(r)/m^2\big) = r/m^2, \tag{5.10}$$

where the function $\check{\mathbf{Y}}_{1,p}(x)$ *is*

$$\check{\mathbf{Y}}_{1,p}(x) = \frac{1}{4}(2\sqrt{x})^{2-1/p}\big(\check{\mathbf{U}}_p\big(\check{\mathbf{T}}_p^{-1}(2\sqrt{x})\big)\big)^{1/p}, \quad x \in \big[\pi^2, \infty\big).$$

We remark from (2.1) that (5.10) is also true for $p = \infty$ by setting

$$\check{\mathbf{Y}}_{1,\infty}(x) := \pi^2 - x, \quad x \in \big[\pi^2, \infty\big).$$

The amplification relation (1.11) for $\mathbf{M}_{m,p}(r)$ can be deduced simply from (5.10), including $p = \infty$ and $p = 1$, as did in the proof of Theorem 1.3.

5.2. *Limiting approach to* $\mathbf{M}_{1,1}(r)$

This subsection is devoted to the proof of Theorem 1.5. That is, we will use the limiting approach to obtain (1.10) and (1.12).

In the following, let $m = 1$, $p \in (1, \infty)$ and $r > 0$. Let

$$\begin{aligned}
\nu_p = \nu_{p,r} &:= \sqrt{\mathbf{M}_{1,p}(r)} \in \big(\sqrt{\pi^2 + r}, \infty\big), \\
a_p = a_{p,r} &:= \check{\mathbf{T}}_p^{-1}(2\nu_{p,r}) \in (0, 1), \\
B_p = B_{p,r} &:= a_{p,r}^{2p^*-2} \in (0, 1).
\end{aligned}$$

We have known

$$\lim_{p\downarrow 1} \nu_{p,r} = \nu_1 = \nu_{1,r} := \sqrt{\mathbf{M}_{1,1}(r)} \in \big(\sqrt{\pi^2 + r}, \infty\big). \tag{5.11}$$

For the maximal values $\mathbf{M}_{1,p}(r)$, some crucial observations are as follows.

Lemma 5.2. *One has the following limits*

$$\lim_{p\downarrow 1} B_p = 1, \tag{5.12}$$

$$\lim_{p\downarrow 1} a_p = 1. \tag{5.13}$$

Proof. Note that (5.13) follows simply from (5.12).

- We assert that

$$\alpha_0 := \liminf_{p\downarrow 1} B_p > 0. \tag{5.14}$$

One has the following elementary inequalities

$$\big(1 - x^2\big)/p^* \leqslant \big(1 - x^{2p^*}\big)/p^* \leqslant 1 - x^2, \quad x \in [0, 1]. \tag{5.15}$$

By (5.6) and (5.15), we have

$$\begin{aligned} 2\nu_p = \check{\mathbf{T}}_p(a_p) &= 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2-B_p(1-x^{2p^*})/p^*}} \\ &\leqslant 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2-B_p(1-x^2)}} \\ &= \frac{2\pi}{\sqrt{1-B_p}}. \end{aligned}$$

Thus

$$B_p \geqslant 1-(\pi/\nu_p)^2 \to 1-(\pi/\nu_1)^2 > 0.$$

See (5.11). Hence we have (5.14).

- We assert that $\alpha_0 = 1$.

Otherwise, let us assume that $\alpha_0 \in (0,1)$. By (5.15), one has

$$\begin{aligned} \frac{1}{\sqrt{1-x^2-B_p(1-x^{2p^*})/p^*}} &\leqslant \frac{1}{\sqrt{1-B_p}\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-\alpha_0+o(1)}}\frac{1}{\sqrt{1-x^2}} \in \mathcal{L}^1. \end{aligned}$$

Moreover, we have, for $x \in [0,1)$,

$$\lim_{p\downarrow 1} \frac{1}{\sqrt{1-x^2-B_p(1-x^{2p^*})/p^*}} = \frac{1}{\sqrt{1-x^2}}.$$

Applying the LDCT to the integral expression (5.6) of $\check{\mathbf{T}}_p(a_p)$, we get

$$\lim_{p\downarrow 1} \check{\mathbf{T}}_p(a_p) = 4\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 2\pi.$$

However, (5.11) shows that this limit shall be $2\nu_1 > 2\pi$. Hence we get a contradiction. This proves the assertion $\alpha_0 = 1$.

Finally, as $B_p \in (0,1)$, we have (5.12) because $\alpha_0 = 1$. $\square$

The limiting equality (5.12) shows that for any $r > 0$, $B_p = B_{p,r}$ is very close to 1 when $p \downarrow 1$. Note that $B_{p,r}$ depends on r as well. Different from (4.3) and (4.10) for minimal values, we are not able to use (5.12) to distinguish $B_{p,r}$ for different r. The approach in the preceding section cannot be adopted to $\mathbf{M}_{1,1}(r)$ in a direct way.

In the following we use a completely different approach to find $\mathbf{M}_{1,1}(r)$. Let $z_p(t) = z_{p,r}(t) := z(t; a_p)$ be the solution of (5.5), where $p \in (1,\infty)$. Note that $z_p(t)$ has different period $2/\nu_p$ for different p. Hence we go back Eqs. (5.2) via transformations (5.4). Set

$$y_p(t) = y_{p,r}(t) := \nu_p^{p-1} z_p(\nu_p t), \quad t \in [0,1].$$

Since $y_p(t)$ is an eigen-function associated with the first Dirichlet eigenvalue, we can take $y_p(t)$ so that

$$y_p(0) = y_p(1) = 0, \qquad y_p(t) > 0 \quad \text{for } t \in (0,1). \tag{5.16}$$

Now the equation for $y_p(t)$ is

$$y_p''(t) + \nu_p^2 y_p(t) - \big(y_p(t)\big)^{2p^*-1} = 0, \quad t \in [0,1]. \tag{5.17}$$

Besides the requirement (3.11) for the L^{2p^*} norm of $y_p(t)$, some properties of $y_p(t)$ are

$$y_p(1-t) \equiv y_p(t) \quad \text{on } [0,1], \tag{5.18}$$

$$\|y_p\|_\infty = y_p(1/2) = \nu_p^{p-1} a_p. \tag{5.19}$$

Our goal is to find the limiting function of $y_p(t)$ as $p \downarrow 1$.

Lemma 5.3. *Let $p_0 > 1$. The set $\{y_p\colon\ 1 < p < p_0\} \subset (W_0^{1,\infty}[0,1], \|\cdot\|')$ is bounded, where $\|y\|' := \|y'\|_\infty$ for $y \in W_0^{1,\infty}[0,1]$.*

Proof. Results (5.11), (5.13) and (5.19) imply

$$\|y_p\|_\infty = \nu_p^{p-1} a_p \to 1 \tag{5.20}$$

as $p \downarrow 1$. The first integral of Eq. (5.17) is

$$\begin{aligned} \big(y_p'(t)\big)^2 + \nu_p^2\big(y_p(t)\big)^2 - \big(y_p(t)\big)^{2p^*}/p^* &= \nu_p^2\big(y_p(1/2)\big)^2 - \big(y_p(1/2)\big)^{2p^*}/p^* \\ &= \nu_p^{2p} a_p^2 (1 - B_p/p^*). \end{aligned} \tag{5.21}$$

In particular, we have from (5.11)–(5.13) and (5.21)

$$\big(y_p'(t)\big)^2 < \nu_p^{2p} a_p^2 + \big(y_p(t)\big)^{2p^*}/p^* \leqslant \nu_p^{2p} a_p^2 + \nu_p^{2p} a_p^2 B_p/p^* \to \nu_1^2.$$

That is, $\|y_p'\|_\infty$ is bounded. $\square$

Let $\{p_n\}$ be any sequence such that $p_n \downarrow 1$. Since $(W_0^{1,\infty}[0,1], \|\cdot\|')$ is compactly embedded into $(\mathbf{C}[0,1], \|\cdot\|_\infty)$, going to a subsequence if necessary, we may assume that

$$y_{p_n} \to y_1 \quad \text{in } \big(\mathbf{C}[0,1], \|\cdot\|_\infty\big). \tag{5.22}$$

Inherited from properties (5.16), (5.18), (5.19) and (5.20), the limiting function y_1 has the following properties.

Lemma 5.4. *The limiting function $y_1 \in \mathbf{C}[0,1]$ satisfies*

$$y_1(0) = y_1(1) = 0, \qquad \|y_1\|_\infty = y_1(1/2) = 1, \tag{5.23}$$

and, for $t \in [0,1]$,

$$y_1(t) \geqslant 0, \qquad y_1(1-t) \equiv y_1(t). \tag{5.24}$$

Due to the symmetry (5.24) of $y_1(t)$, we need only to consider $y_1(t)$ for $t \in I_0 := [0, 1/2]$.

Lemma 5.5. *For $t \in I_0$, $y_1(t)$ is differentiable and satisfies the following limiting equation*

$$y_1' = \nu_1 \sqrt{1 - y_1^2} =: f(y_1). \tag{5.25}$$

Proof. For $p \in (1, \infty)$ and $t \in I_0$, Eq. (5.21) reads as

$$y_p'(t) = \sqrt{\nu_p^{2p} a_p^2 - \nu_p^2 \bigl(y_p(t)\bigr)^2 + g_p(t)}, \tag{5.26}$$

where

$$g_p(t) := \bigl(y_p(t)\bigr)^{2p^*}/p^* - \nu_p^{2p} a_p^2 B_p/p^*.$$

By boundary conditions (5.16) and Eq. (5.26),

$$y_p(t) = \int_0^t \sqrt{\nu_p^{2p} a_p^2 - \nu_p^2 \bigl(y_p(t)\bigr)^2 + g_p(t)}\,\mathrm{d}t, \quad t \in I_0. \tag{5.27}$$

By Lemma 5.2 and (5.20), (5.22), as $p_n \downarrow 1$, we have $\nu_{p_n}^{2p_n} a_{p_n}^2 \to \nu_1^2$ and

$$\nu_{p_n}^2 \bigl(y_{p_n}(t)\bigr)^2 \to \nu_1^2 \bigl(y_1(t)\bigr)^2, \qquad \bigl|g_{p_n}(t)\bigr| \leqslant 2\nu_{p_n}^{2p_n} a_{p_n}^2 \mathbf{b}_{p_n}/p_n^* \to 0.$$

The latter two are uniform in $t \in I_0$. Let $p = p_n$ in (5.27) and $n \to \infty$. We obtain the equality

$$y_1(t) = \int_0^t \nu_1 \sqrt{1 - y_1^2(t)}\,\mathrm{d}t, \quad t \in I_0.$$

This shows that $y_1 \in \mathbf{C}^1(I_0)$ and satisfies the nonlinear ODE (5.25). $\square$

In the following we will use ODE (5.25) to find the limiting function $y_1(t)$, $t \in I_0$. Note first from (5.25) that $y_1(t)$ is non-decreasing in $t \in I_0$. Moreover, $y_1(t)$ satisfies the boundary conditions in (5.23). One crucial observation on ODE (5.25) is that it is singular at $y_1 = \pm 1$. That is, $f(y_1)$ is not differentiable at $y_1 = \pm 1$. Hence the solutions of (5.25) satisfying $y_1(1/2) = 1$ are not unique. In fact, besides the constant solution $y(t) \equiv 1$, problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \nu_1 \sqrt{1 - y^2}, \quad t \in \mathbb{R}, \qquad y(1/2) = 1$$

has also the following family of solutions

$$\varphi_\alpha(t) := \begin{cases} -1 & \text{for } t \in (-\infty, \alpha - \pi/\nu_1], \\ \cos(\nu_1(t - \alpha)) & \text{for } t \in [\alpha - \pi/\nu_1, \alpha], \\ 1 & \text{for } t \in [\alpha, \infty), \end{cases} \tag{5.28}$$

where the parameter $\alpha \in (-\infty, 1/2]$.

For our limiting function $y_1(t)$, $t \in I_0$, let us define

$$\alpha = \min\{t \in I_0:\ y_1(t) = 1\} \in (0, 1/2].$$

Then $y_1(t) \equiv \varphi_\alpha(t)$ for $t \in I_0$, where $\varphi_\alpha(t)$ is as in (5.28). Since $y_1(t) \geqslant 0$ for $t \in I_0$, it is easy to see from (5.28) that $\alpha \in [0, \pi/(2\nu_1)]$. Moreover, $y_1(0) = \varphi_\alpha(0) = 0$ shows that

$$\alpha = \pi/(2\nu_1) \in (0, 1/2), \tag{5.29}$$

by recalling $\nu_1 > \pi$. From (5.28) and (5.29), the limiting function $y_1(t)$ is uniquely determined by ν_1. That is,

$$y_1(t) = \begin{cases} \sin(\nu_1 t) & \text{for } t \in [0, \pi/(2\nu_1)], \\ 1 & \text{for } t \in [\pi/(2\nu_1), 1/2]. \end{cases} \tag{5.30}$$

From Eq. (5.26), one has also

$$y'_{p_n}(t) \to \nu_1\sqrt{1 - \big(y_1(t)\big)^2} = y'_1(t) = \begin{cases} \nu_1 \cos(\nu_1 t) & \text{for } t \in [0, \pi/(2\nu_1)], \\ 0 & \text{for } t \in [\pi/(2\nu_1), 1/2], \end{cases} \tag{5.31}$$

uniformly in $t \in I_0$. Since the limits (5.30) and (5.31) are independent of sequences $p_n \downarrow 1$, we conclude the following result.

Lemma 5.6. *As $p \downarrow 1$, one has*

$$y_p \to y_1 \quad \text{in } \mathbf{C}^1(I_0), \tag{5.32}$$

where $y_1(t) = y_{1,r}(t)$ is given by (5.30).

Proof of Theorem 1.5. Multiplying Eq. (5.17) by $y_p(t)$ and integrating over $[0, 1]$, we obtain

$$\|y_p\|_{2p^*}^{2p^*} = \int_0^1 \big(\nu_p^2 y_p^2 + y_p y''_p\big)\,\mathrm{d}t = \nu_p^2 \|y_p\|_2^2 - \|y'_p\|_2^2,$$

because $y_p(0) = y_p(1) = 0$. Using equality (3.11), we get

$$\nu_p^2 \|y_p\|_2^2 - \|y'_p\|_2^2 = \|y_p\|_{2p^*}^{2p^*} = r^p.$$

By the symmetry (5.18), we get

$$2\nu_p^2 \|y_p\|^2_{L^2[0,1/2]} - 2\|y'_p\|^2_{L^2[0,1/2]} = r^p.$$

By letting $p \downarrow 1$, we use the uniform convergence (5.32) to obtain the limiting equality

$$2\nu_1^2 \|y_1\|^2_{L^2[0,1/2]} - 2\|y'_1\|^2_{L^2[0,1/2]} = r.$$

By formulas (5.30) and (5.31), we have

$$2\nu_1^2 \|y_1\|^2_{L^2[0,1/2]} - 2\|y'_1\|^2_{L^2[0,1/2]} = \big(\nu_1^2 - \pi\nu_1/2\big) - \pi\nu_1/2 = \nu_1^2 - \pi\nu_1.$$

Hence $\nu_1 > \sqrt{\pi^2 + r}$ satisfies

$$\nu_1^2 - \pi \nu_1 = r,$$

with the unique solution being

$$\mathbf{M}_{1,1}(r) = \nu_1^2 = \left(\pi + \sqrt{\pi^2 + 4r}\right)^2/4 = \mathbf{Y}_1(r).$$

This is the desired result (1.12). $\square$

Remark 5.7. We remark that (1.12) is consistent with the known lower bound (5.3) for $\mathbf{M}_{1,1}(r)$. Moreover, if one introduces the function

$$\mathbf{Y}_{1,1}(x) = x - \pi\sqrt{x}, \quad x \in \left[\pi^2, \infty\right),$$

formula (5.10) is also true for $p = 1$.

Corollary 5.8. *Let $r \geqslant 0$. The maximal value $\mathbf{M}_{1,1}(r) = \mathbf{Y}_1(r)$ in $B_1[r]$ can be realized by the following potential $q_{1,r} \in S_1[r] \subset B_1[r]$*

$$q_{1,r}(t) := \begin{cases} 0 & \text{for } t \in [0, \pi/\mathbf{c}_r) \cup (1 - \pi/\mathbf{c}_r, 1], \\ -\mathbf{Y}_1(r) & \text{for } t \in [\pi/\mathbf{c}_r, 1 - \pi/\mathbf{c}_r], \end{cases}$$

where

$$\mathbf{c}_r := 2\sqrt{\mathbf{Y}_1(r)} = \pi + \sqrt{\pi^2 + 4r}.$$

That is,

$$\mathbf{M}_{1,1}(r) = \max\left\{\lambda_1(q)\colon q \in B_1[r]\right\} = \lambda_1(q_{1,r}) = \mathbf{Y}_1(r).$$

Moreover, the corresponding eigen-function is $y_{1,r}(t)$ given by (5.30) on $[0, 1/2]$ and $y_{1,r}(t) = y_{1,r}(1-t)$ for $t \in [1/2, 1]$.

Proof. These can be verified directly. $\square$

Remark 5.9. Corollary 5.8 shows that the maximal problems $\mathbf{M}_{1,1}(r)$ and $\mathbf{M}_{m,1}(r)$ in L^1 balls $B_1[r]$ are completely different from the corresponding minimal problems $\mathbf{L}_{1,1}(r)$ and $\mathbf{L}_{m,1}(r)$, because the latter problems have no minimizers in $B_1[r]$ for all $r > 0$. See Remark 4.7. This is why the approach in the preceding sections for $\mathbf{L}_{m,1}(r)$ cannot be applied to $\mathbf{M}_{m,1}(r)$ in a direct way.

6. Extremal values of other eigenvalues

Given $q \in \mathcal{L}^p$. Problem (1.1) with the Neumann boundary condition $x'(0) = x'(1) = 0$ has also a sequence of eigenvalues

$$\lambda_0^N(q) < \lambda_1^N(q) < \cdots < \lambda_m^N(q) < \cdots.$$

For $q \in \mathcal{L}^p = L^p([0,1], \mathbb{R})$, $q(t)$ can be extended to $\mathbb{R}$ by 1-periodicity. In this sense one can identify $\mathcal{L}^p$ as $L^p(\mathbb{S}_1, \mathbb{R})$, $\mathbb{S}_1 = \mathbb{R}/\mathbb{Z}$. It is well known that problem (1.1) defines also a double-sequence

$$\bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leqslant \bar{\lambda}_1(q) < \cdots < \underline{\lambda}_m(q) \leqslant \bar{\lambda}_m(q) < \cdots$$

such that $\underline{\lambda}_m(q)$, $\bar{\lambda}_m(q)$ are 1-periodic eigenvalues of (1.1) for m even, and $\underline{\lambda}_m(q)$, $\bar{\lambda}_m(q)$ are 1-anti-periodic eigenvalues of (1.1) for m odd. See [8,19].

Recall from [19, Theorem 4.3] the following relations between Dirichlet, Neumann and periodic, anti-periodic eigenvalues

$$\underline{\lambda}_m(q) = \min\{\lambda_m^\sigma(q_s):\ s \in [0,1]\}, \qquad \bar{\lambda}_m(q) = \max\{\lambda_m^\sigma(q_s):\ s \in [0,1]\} \tag{6.1}$$

for all $q \in L^p(\mathbb{S}_1)$ and all $m \in \mathbb{N}$. Here $\sigma = N$ or D, $q_s(t) := q(t+s)$ are translations of $q(t)$. Notice that, in preceding sections, $\lambda_m^D(q)$ is written as $\lambda_m(q)$. Due to relations like (6.1), the extremal values $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ of Dirichlet eigenvalues are also actually the extremal values for other related eigenvalues.

For 1-periodic and 1-anti-periodic eigenvalues $\underline{\lambda}_m(q)$ and $\bar{\lambda}_m(q)$ of (1.1), we have the following four sequences of extremal values

$$\underline{\mathbf{L}}_{m,p}(r) := \inf_{q \in B_p[r]} \underline{\lambda}_m(q), \qquad \bar{\mathbf{L}}_{m,p}(r) := \inf_{q \in B_p[r]} \bar{\lambda}_m(q), \tag{6.2}$$

$$\underline{\mathbf{M}}_{m,p}(r) := \sup_{q \in B_p[r]} \underline{\lambda}_m(q), \qquad \overline{\mathbf{M}}_{m,p}(r) := \sup_{q \in B_p[r]} \bar{\lambda}_m(q), \tag{6.3}$$

where $m \in \mathbb{Z}^+$, $p \in [1,\infty]$ and $r \geqslant 0$. Note that, for $m = 0$, $\underline{\mathbf{L}}_{0,p}(r)$ and $\underline{\mathbf{M}}_{0,p}(r)$ are void. For the zeroth periodic eigenvalues $\bar{\lambda}_0(q)$, one has $\overline{\mathbf{M}}_{0,p}(r) \equiv r$, while

$$\bar{\mathbf{L}}_{0,1}(r) \equiv \mathbf{Z}_0^{-1}(r),$$

where

$$\mathbf{Z}_0(x) := 2\sqrt{-x}\tanh(\sqrt{-x}/2), \quad x \in (-\infty, 0].$$

See (1.4) and Theorem 1.2 of [22] respectively. For $m \in \mathbb{N}$, it follows from relations (6.1) and the results for Dirichlet eigenvalues that all extremal values of (6.2)–(6.3) are well defined.

For $m \in \mathbb{N}$, two extremal values $\underline{\mathbf{L}}_{m,p}(r)$ and $\overline{\mathbf{M}}_{m,p}(r)$ of (6.2)–(6.3) can be reduced to $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ for the Dirichlet eigenvalues.

Theorem 6.1. *For all $m \in \mathbb{N}$, $p \in [1,\infty]$, $r \geqslant 0$, there hold the following equalities*

$$\underline{\mathbf{L}}_{m,p}(r) = \mathbf{L}_{m,p}(r), \qquad \overline{\mathbf{M}}_{m,p}(r) = \mathbf{M}_{m,p}(r). \tag{6.4}$$

In particular, $\underline{\mathbf{L}}_{1,1}(r)$ is just $\mathbf{Z}_1^{-1}(r)$ and $\overline{\mathbf{M}}_{1,1}(r)$ is just $\mathbf{Y}_1(r)$.

Proof. We use the Dirichlet eigenvalues in relations (6.1). Thus

$$\underline{\lambda}_m(q) \leqslant \lambda_m^D(q) \leqslant \bar{\lambda}_m(q).$$

Taking the infimum and supremum over $q \in B_p[r]$ respectively, we get

$$\underline{\mathbf{L}}_{m,p}(r) \leqslant \mathbf{L}_{m,p}(r) \leqslant \bar{\mathbf{L}}_{m,p}(r), \qquad \underline{\mathbf{M}}_{m,p}(r) \leqslant \mathbf{M}_{m,p}(r) \leqslant \overline{\mathbf{M}}_{m,p}(r), \tag{6.5}$$

where $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ are extremal values of Dirichlet eigenvalues.

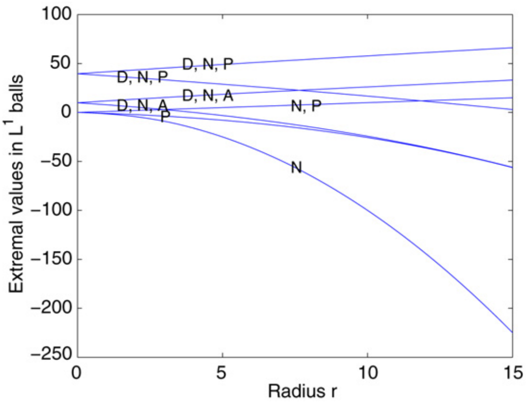

Fig. 4. Extremal values of the mth eigenvalues with potentials in L^1 balls where $m = 0, 1, 2$.

On the other hand, for any given $q \in B_p[r]$, we have some $t_0 \in \mathbb{R}$ such that

$$\underline{\lambda}_m(q) = \lambda_m^D(q_{t_0}) \geqslant \mathbf{L}_{m,p}(r),$$

because $\|q_{t_0}\|_p = \|q\|_p$. Taking the infimum, we have

$$\underline{\mathbf{L}}_{m,p}(r) \geqslant \mathbf{L}_{m,p}(r). \tag{6.6}$$

Now the first equality of (6.4) follows from (6.5) and (6.6). The second equality of (6.4) can be obtained in a similar way. □

Instead of the Dirichlet eigenvalues, we can use the Neumann eigenvalues in the proof of Theorem 6.1. Thus (6.4) is also true when $\mathbf{L}_{m,p}(r)$ and $\mathbf{M}_{m,p}(r)$ are replaced by the corresponding extremal values $\mathbf{L}_{m,p}^N(r)$ and $\mathbf{M}_{m,p}^N(r)$ of the Neumann eigenvalues $\lambda_m^N(q)$, $m \in \mathbb{N}$. From these, we have the following results.

Corollary 6.2. *Let $m \in \mathbb{N}$, $p \in [1, \infty]$, $r \geqslant 0$. One has*

$$\mathbf{L}_{m,p}^N(r) = \mathbf{L}_{m,p}(r), \qquad \mathbf{M}_{m,p}^N(r) = \mathbf{M}_{m,p}(r).$$

In summary, combined with the results of [22], for eigenvalues of (1.1) with potentials in L^1 balls, we have obtained the following explicit, elementary extremal functions.

- $\mathbf{L}_{0,1}^N(r) = \hat{\mathbf{Z}}_0^{-1}(r)$, the minimal values of the zeroth Neumann eigenvalues,
- $\bar{\mathbf{L}}_{0,1}(r) = \mathbf{Z}_0^{-1}(r)$, the minimal values of the zeroth periodic eigenvalues,
- $\mathbf{L}_{m,1}(r) = m^2\mathbf{Z}_1^{-1}(r/m^2)$, $m \in \mathbb{N}$, the minimal values of the Dirichlet, Neumann and periodic, anti-periodic eigenvalues, abbreviated as D, N, P and A eigenvalues, respectively,
- $\mathbf{M}_{0,1}(r) = r$, the maximal values of the zeroth N and the zeroth P eigenvalues, and
- $\mathbf{M}_{m,1}(r) = m^2\mathbf{Y}_1(r/m^2)$, $m \in \mathbb{N}$, the maximal values of the D, N, P and A eigenvalues.

These extremal functions are plotted in Fig. 4.

Let us mention some open problems of extremal values for periodic/anti-periodic eigenvalues. For the zeroth periodic eigenvalues $\bar{\lambda}_0(q)$, both $\bar{\mathbf{L}}_{0,p}(r)$ and $\overline{\mathbf{M}}_{0,p}(r)$ are clear [22]. However, for $m \in \mathbb{N}$, we have only known $\underline{\mathbf{L}}_{m,p}(r)$ and $\overline{\mathbf{M}}_{m,p}(r)$ in Theorem 6.1. The following are open.

Problem 6.3. Let $m \in \mathbb{N}$. What are the characterizations of extremal values $\bar{\mathbf{L}}_{m,p}(r)$ and $\underline{\mathbf{M}}_{m,p}(r)$? Whether $\bar{\mathbf{L}}_{m,1}(r)$ and $\underline{\mathbf{M}}_{m,1}(r)$ can be found explicitly?

Let us see why these are open. Since $\bar{\lambda}_0(q)$ is simple, $\bar{\lambda}_0 : (\mathcal{L}^p, \|\cdot\|_p) \to \mathbb{R}$ is also continuously differentiable. The extremal values of $\bar{\lambda}_0(q)$ can be studied similarly, because $\bar{\lambda}_0(q)$ is also continuous in $q \in (\mathcal{L}^p, w_p)$ [21]. In fact, in [22], we first use these approaches to solve the extremal problems for $\bar{\lambda}_0(q)$ and then reduce $\lambda_0^N(q)$ to problems of $\bar{\lambda}_0(q)$ by a simple scaling technique. However, when $m \in \mathbb{N}$, the pair of eigenvalues $\underline{\lambda}_m(q)$ and $\bar{\lambda}_m(q)$ may coexist, i.e.,

$$\underline{\lambda}_m(q) = \bar{\lambda}_m(q). \tag{6.7}$$

See [1,3]. At those q so that coexistence (6.7) occurs, both $\underline{\lambda}_m(q)$ and $\bar{\lambda}_m(q)$ may not be continuously differentiable at q. See [6,9,18]. Though the minimizers for $\bar{\mathbf{L}}_{m,p}(r)$ and maximizers for $\underline{\mathbf{M}}_{m,p}(r)$ exist when $p \in (1, \infty)$, the variational approach cannot be applied in a direct way because of the lack of continuous differentiability. Notice that the coexistence (6.7) is the most delicate problem for linear equations with periodic coefficients.

Finally, let us mention some extremal problems of weighted Dirichlet eigenvalues of

$$x'' + \lambda\rho(t)x = 0, \quad \rho(t) \geqslant 0, \ t \in [0, 1].$$

See [5,7,16]. In a classical paper [7], Krein has completely solved the extremal values of the weighted Dirichlet eigenvalues with the assumption on densities $\rho(t)$

$$0 \leqslant \rho(t) \leqslant h < \infty \quad \text{a.e. } t, \qquad \|\rho\|_1 = r.$$

The approach there is completely different from here. In [17], Yan and Zhang have generalized the results in [21] to eigenvalues of the one-dimensional p-Laplacian. Some generalized problems of Krein can be solved using the analytical method developed in [22] and in this paper. We think the approaches in [22] and the present paper are more analytical and have applications to other extremal problems of eigenvalues.

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